Solution of Partial Integro-Differential Equations by using Laplace, Elzaki and Double Elzaki Transform Methods

Mrs. Gore(Jagtap) Jyotsana S.[1] , Mr. Gore Shukracharya S.[2]

[1] Lecturer, Mathematics Department, MET'S IOE-Nashik, Maharashtra,India.
[2] Assistant Professor, E &TC Department, MCOE & RC-Nashik, Maharashtra,India.

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Abstract - *Partial integro–differential equations (PIDE) occur in several fields of sciences and mathematics. The main purpose of this paper to study how to solve partial integro–differential equation (PIDE) by using various methods like Laplace, Elzaki and Double Elzaki Transform.*

To solve PIDE by using Laplace Transform (LT), first convert Proposed PIDE to an ordinary differential equation (ODE) then solving this ODE by applying inverse LT we get an exact solution of the problem.

To solve PIDE by using Elzaki Transform (ET), first convert Proposed PIDE to an ordinary differential equation (ODE) then solving this ODE by applying inverse ET we get an exact solution of the problem.

To solve PIDE by using Double Elzaki Transform (DET), first convert Proposed PIDE to an algebraic equation , Solving this algebraic equation & applying double inverse Elzaki transform we obtain the exact solution of the problem.

These methods are useful tools for the solution of the differential and integral equation and linear system of differential and integral equation.

Key Words: Partial integro–differential equations (PIDE), Ordinary differential equation (ODE), Laplace Transform (LT), Elzaki Transform (ET), Double Elzaki Transform (DET).

1. INTRODUCTION

In the last few years theory and application of partial integro – differential equation (PIDE) play an important role in the various fields of many problems of mathematical fields, engineering physics, biology, and social sciences [4-15]. This explains a growing interest in the mathematics community to integro-differential equations and in particular to partial integro-differential equations.

Therefore it is very important to know various methods to solve such partial differential equations [1-3].

One tool for solving linear PIDE's is the Laplace transform (LT) method [1]. LT is used for calculations of water flow and heat transfer in fractured rocks [1].

Second tool for solving linear PIDE's is Elzaki transform method. It is one of the useful tool for solution of the differential, integral equation and linear system of differential and integral equation [2].

Third tool is Double Elzaki Transform which is the higher version of Elzaki Transform to solve linear PIDE's [3].

In this paper we solve single example of PIDE by using three different methods like *Laplace, Elzaki and Double Elzaki Transform.*

2. PRELIMINARIES

2.1 Laplace Transform method:

Definition: The Laplace transform of a function f(x), is defined by

$$\bar{f}(p) = L[f(x)] = \int_0^t e^{-px} f(x)\, dx \ ; \ x \geq 0$$

(Whenever integral on RHS exists)
Where, $x \geq 0$, p is real and L is the Laplace transform operator.

Convolution Theorem:

Let f(t) and g(t) having Laplace transform $L[f(t)]$ and $L[g(t)]$,

i.e If $\bar{f}(p) = L[f(t)]$ *and* $\bar{g}(p) = L[g(t)]$

then Laplace transform of the convolution of f and g ,

$$f(t) * g(t) = \int_0^t f(x-t) g(t) dt$$

is given by ,

$$L[f(t) * g(t)] = L[f(t)] . L[g(t)] = \bar{f}(p) . \bar{g}(p)$$

Solving PIDEs using Laplace Transform Method:

Consider general linear PIDE,

$$\sum_{i=0}^{m} a_i \frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i(t-s) \frac{\partial^i u(x,s)}{\partial x^i}\, ds + f(x,t) = 0$$
..............(1)

(with prescribed condition)
Where $f(x,t)$ and $k_i(t,s)$ are known functions.
$a_i's, b_i's, d_i's$ and c are constants or the functions of x.
Taking Laplace transform on both sides of PIDE(1) with respect to t we get,

$$\sum_{i=0}^{m} a_i \, L\left\{\frac{\partial^i u}{\partial t^i}\right\} \sum_{i=0}^{n} b_i \, L\left\{\frac{\partial^i u}{\partial x^i}\right\} + cL\{u\} + \sum_{i=0}^{r} d_i \, L\left\{k_i(t) * \frac{\partial^i u(x,t)}{\partial x^i}\right\} + L\{f(x,t)\} = 0$$

Using convolution theorem for Laplace transform we get

$$\sum_{i=0}^{m} a_i \left(p^i \bar{u}(x,p) - \sum_{j=1}^{i} \left(p^{j-1} u^{(i-j)}(x,0)\right)\right) + \sum_{i=0}^{n} b_i \frac{d^i \bar{u}(x,p)}{dx^i} + c\,\bar{u}(x,p) + \sum_{i=0}^{r} d_i \, \bar{k}_i(p) \frac{d^i \bar{u}(x,p)}{dx^i} + \bar{f}(x,p) = 0$$
.................(2)

Where, $\bar{u}(x,p) = L\{u(x,t)\}$.

$\bar{f}(x,p) = L\{f(x,t)\}$.

and $\bar{k}_i(p) = L\{k_i(t)\}$.

Equation (2) is an ordinary differential equation in ū(x,p). Solving this ODE and taking inverse Laplace transform of ū(x,p), we get a solution u(x,t) of (1).

Illustrative example:

Example.
Consider the PIDE

$$u_{tt} = u_x + 2\int_0^t (t-s).u(x,s)\, ds - 2e^x \qquad \text{.......... (3)}$$

With initial condition

$$u(x,0) = e^x \quad , \; u_t(x,0) = 0 \qquad \text{.........(4)}$$

And boundary condition

$$u(0,t) = \cos t \qquad \text{.........(5)}$$

Solution:-

Taking Laplace transform w.r.to t on both sides of (3)

$$p^2\bar{u}(x,p) - pu(x,0) - u_t(x,0) = \frac{d\bar{u}}{dx} + 2\left(\frac{1}{p^2}\right)\bar{u} - 2e^x\left(\frac{1}{p}\right)$$

$$\therefore \quad \frac{d\bar{u}}{dx} + \left(\frac{2}{p^2} - p^2\right)\bar{u} = \frac{2}{p} - p \qquad \text{...........(6)}$$

$$\bar{u}(x,p) = \frac{1}{e^{\int\left(\frac{2}{p^2}-p^2\right)dx}}\left[\int e^{\int\left(\frac{2}{p^2}-p^2\right)dx}.\left(\frac{2}{p} - p\right)dx + C\right]$$

Therefore the solution of (6) is

$$\bar{u}(x,p) = \left(\frac{p}{p^2+1}\right)e^x + Ce^{\left(p^2 - \frac{2}{p^2}\right)x} \qquad \text{............(7)}$$

From the boundary condition (5)

$$\bar{u}(0,p) = \frac{p}{p^2+1} \qquad \text{...............(8)}$$

Using (7) and (8) we get C=0
∴ Equation (7) becomes,

$$\bar{u}(x,p) = \left(\frac{p}{p^2+1}\right)e^x$$
...............(9)

Applying inverse Laplace transform for (9), we get exact solution
$u(x,t) = e^x \cos t$.

2. 2 Elzaki Transform

Definition:

Let a function f(t) defined for $t > 0$ then Elzaki transform of f(t) is the function T defined as follows,

$$E[f(t)] = T(v) = v\int_0^\infty f(t)\, e^{-\frac{t}{v}}\, dt \quad , \quad t > 0$$

Theorem -1

Elzaki transform of partial derivatives are in the form

1) $E\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{v}T(x,v) - v\,f(x,0)$

2) $E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - f(x,0) - v\,\frac{\partial f(x,0)}{\partial t}$

3) $E\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d\,[T(x,v)]}{dx}$

4) $E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2\,[T(x,v)]}{dx^2}$

Theorem -2 (Convolution):

Let f(t) and g(t) having Elzaki transform M(v) and N(v) , then Elzaki transform of the convolution of f and g ,

$f(t) * g(t) = \int_0^\infty f(t) . g(t-\tau)\, d\tau$, is given by:

$$E[f(t) * g(t)] = \frac{1}{v} M(v)\, N(v)$$

Solving PIDEs using Elzaki Transform Method :

Consider general linear PIDE,

$$\sum_{i=0}^{m} a_i \frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i(t-s) \frac{\partial^i u(x,s)}{\partial x^i} ds + f(x,t) = 0 \qquad \text{............(10)}$$

(with prescribed condition)

Where $f(x,t)$ and $k_i(t,s)$ are known functions. $a_i's$, $b_i's$, $d_i's$ and c are constants or the functions of x.

Taking Elzaki transform on both sides of PIDE(10) with respect to t we get,

$$\sum_{i=0}^{m} a_i\ E\left\{\frac{\partial^i u}{\partial t^i}\right\} + \sum_{i=0}^{n} b_i\ E\left\{\frac{\partial^i u}{\partial x^i}\right\} + cE\{u\} + \sum_{i=0}^{r} d_i\ E\left\{\int_0^t k_i(t-s) \frac{\partial^i u(x,s)}{\partial x^i} ds\right\} + E\{f(x,t)\} = 0$$

Using theorem 1 and theorem 2 for Elzaki transform we get,

$$\sum_{i=0}^{m}\left[a_i \frac{\bar{u}(x,v)}{v^i} - \sum_{k=0}^{i-1} v^{2-i+k}\, .u^{(i-k)}(x,0)\right] + \sum_{i=0}^{n} b_i \frac{d^i \bar{u}(x,v)}{dx^i} + c\bar{u}(x,v) + \sum_{i=0}^{r} d_i \frac{1}{v} \bar{k}_i(v) \frac{d^i \bar{u}(x,v)}{dx^i} + \bar{f}(x,v) = 0 \qquad \text{............(11)}$$

Where $\bar{u}(x,v) = E[u(x,v)]$, $\bar{f}(x,v) = E[f(x,v)]$,

$\bar{k}_i(v) = E[k_i(t)]$

Equation (11) is an ordinary differential equation in ū(x,v). Solving this ODE and taking inverse Elzaki transform of ū(x,v), we get a solution u(x,t) of (10).

Illustrative example:

Example .

Consider the PIDE

$$u_{tt} = u_x + 2\int_0^t (t-s). u(x,s)\, ds - 2e^x \qquad \text{......(12)}$$

With initial conditions

$$u(x,0) = e^x \ ,\ u_t(x,0) = 0 \qquad \text{........(13)}$$

And boundary condition

$$u(0,t) = \cos t \qquad \text{........(14)}$$

Solution:-

Taking Elzaki transform w.r.to t on both sides of (12)

$$\frac{\bar{u}(x,v)}{v^2} - u(x,0) - v\, u_t(x,0) = \bar{u}_x + \frac{1}{v}(2v^3 . \bar{u}) - 2e^x v^2.$$

$$\bar{u}_x + 2v^2 . \bar{u} - \frac{u}{v^2} - 2e^x v^2 + e^x = 0.$$

$$\frac{d\bar{u}}{dx} + \left(2v^2 - \frac{1}{v^2}\right)\bar{u} = e^x(2v^2 - 1). \qquad \text{......(15)}$$

Therefore the solution of (15) is,

$$\bar{u} = \frac{v^2}{v^2+1} e^x + C\, e^{-\left(2v^2 - \frac{1}{v^2}\right)x} \qquad \text{..........(16)}$$

From boundary condition (14),

$$\bar{u}(0,v) = E\{u(0,t)\} = E\{\cos t\} = \frac{v^2}{v^2+1}. \qquad \text{............(17)}$$

Using (16) and (17), to get

$$C = 0. \qquad \text{..........(18)}$$

Then equation (11) becomes,

$$\bar{u} = \frac{v^2}{v^2+1} e^x \qquad \text{............(19)}$$

Applying inverse Elzaki transform on both side of (19) , we get exact solution ,

$$u(x,t) = E^{-1}\{\bar{u}(x,v)\} = e^x\, E^{-1}\left\{\frac{v^2}{v^2+1}\right\}$$

$$\therefore u(x,t) = e^x \cos t. \qquad \text{..........(20)}$$

2.3 Double Elzaki Transform

Definition:

Let $f(x,t)$, where $t, x \in R^+$ be a function which can be expressed as a convergent infinite series then, its double Elzaki transform, given by

$$E_2[f(x,t), u, v] = T(u,v) = uv \int_0^\infty \int_0^\infty f(x,t)\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} dx\, dt\ ,\quad x,t > 0. \qquad \text{...........(21)}$$

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Where u & v are complex values.

Theorem -1

Double Elzaki transform of first and second order partial derivatives are in the form ,

1) $E_2\left\{\frac{\partial f}{\partial x}\right\} = \frac{1}{u}T(u,v) - u\,T(0,v)$.

2) $E_2\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \frac{1}{u^2}T(u,v) - T(0,v) - u\,\frac{\partial\,[T(0,v)]}{\partial x}$.

3) $E_2\left\{\frac{\partial f}{\partial t}\right\} = \frac{1}{v}T(u,v) - v\,T(u,0)$.

4) $E_2\left\{\frac{\partial^2 f}{\partial t^2}\right\} = \frac{1}{v^2}T(u,v) - T(u,0) - v\,\frac{\partial\,[T(u,0)]}{\partial t}$.

5) $E_2\left\{\frac{\partial^2 f}{\partial x\,\partial t}\right\} = \frac{1}{v}T(u,v) - v\,T(u,0) - \frac{u}{v}T(0,v) + \mathrm{uv}\,\mathrm{T}(0,0)$.

Theorem -2 (Convolution):

Let $f(x,t)$ and $g(x,t)$ be the functions having Double Elzaki transform $M(u,v)$ and $N(u,v)$ then the Double Elzaki transform of the convolution of $f(x,t)$ and $g(x,t)$ is,

$$E_2[(f*g)(x,t);(u,v)] = \frac{1}{uv}\,M(u,v)\,N(u,v) \quad \text{..........(22)}$$

Solving **PIDE's** using Double Elzaki Transform Method :

Consider the general linear partial integro-differential equation,

$$\sum_{i=0}^{m} a_i\,\frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} b_i\,\frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^{r} d_i\,\int_0^t k_i(t-s)\,\frac{\partial^i u(x,s)}{\partial x^i}\,ds + f(x,t) = 0 \quad \text{............(23)}$$

(with prescribed condition)

Where $f(x,t)$ *and* $k_i(t,s)$ are known functions. a_i, b_i, d_i *and* c are constants or the functions of x.

Taking double Elzaki transform on both sides of PIDE(23) with respect to t we get,

$$\sum_{i=0}^{m} a_i\;E_2\left\{\frac{\partial^i u}{\partial t^i}\right\} + \sum_{i=0}^{n} b_i\;E_2\left\{\frac{\partial^i u}{\partial x^i}\right\} + cE_2\{u\} + \sum_{i=0}^{r} d_i\;E_2\left\{\int_0^t k_i(t-s)\,\frac{\partial^i u(x,s)}{\partial x^i}\,ds\right\} + E_2\{f(x,t)\} = 0 \quad \text{..............(24)}$$

Using theorem 1 and theorem 2 for double Elzaki transform we get,

$$\sum_{i=0}^{m} a_i\;\left\{\frac{\bar{u}(x,v)}{v^i} - \sum_{k=0}^{i-1} v^{2-i+k}\,E_x\left\{\frac{\partial^k \bar{u}(x,0)}{\partial t^k}\right\}\right\} + \sum_{i=0}^{n} b_i\;\left\{\frac{\bar{u}(x,v)}{v^i} - \sum_{j=0}^{i-1} u^{2-i+j}\,E_t\left\{\frac{\partial^j \bar{u}(0,t)}{\partial x^j}\right\}\right\} + c\bar{u}(x,v) + \sum_{i=0}^{r} d_i\;\frac{1}{v}\bar{k}_i(v)\left\{\frac{\bar{u}(x,v)}{u^i} - \sum_{j=0}^{i-1} u^{2-i+j}\,E_t\left\{\frac{\partial^j \bar{u}(0,t)}{\partial x^j}\right\}\right\} + \bar{f}(x,v) = 0. \quad \text{..............(25)}$$

Where $\bar{u}(x,v) = E_2[u(x,v)]$, $\bar{f}(x,v) = E_2[\bar{f}(x,v)]$,

$\bar{k}_i(v) = E_2[\bar{k}_i(t)]$

Equation (25) is an algebraic equation in $\bar{u}(x,v)$. Solving algebraic equation and take inverse double Elzaki transform of we get, $\bar{u}(x,v)$, we get an exact solution $u(x,t)$ of (2).

Illustrative example:

<u>Example</u> .

Consider the PIDE

$$u_{tt} = u_x + 2\int_0^t (t-s).u(x,s)\,ds - 2e^x \quad \text{.......(26)}$$

With initial condition

$$u(x,0) = e^x \quad ,\; u_t(x,0) = 0 \quad \text{..........(27)}$$

And boundary condition

$$u(0,t) = cost \quad \text{...........(28)}$$

<u>Solution</u>:-

Taking double Elzaki transform of equation (26)

$$\frac{1}{v^2}T(u,v) - T(u,0) - v\,T_t(u,0) = \frac{1}{u}T(u,v) - uT(0,v) + 2v^2T(u,v) - 2\,\frac{u^2v^2}{1-u} \quad \text{..........(29)}$$

And single Elzaki transforms of initial condition (27) & boundary condition (28) are given by

$$T(u,0) = \frac{u^2}{1-u}, \quad T_t(u,0) = 0, \quad T(0,v) = \frac{v^2}{1-u}$$

Then equation (29) becomes

$$\frac{1}{v^2}T(u,v) - \frac{u^2}{1-u} = \frac{1}{u}T(u,v) - \frac{uv^2}{1-u} + 2v^2T(u,v) - 2\,\frac{u^2v^2}{1-u}.$$

$$\left(\frac{1}{u} + 2v^2 - \frac{1}{v^2}\right) T(u,v) = \frac{uv^2}{1+v^2} - \frac{u^2}{1-u} + 2\frac{u^2v^2}{1-u}.$$

$$\left(\frac{v^2+2uv^4-u}{uv^2}\right) T(u,v) = \frac{(1-u)uv^2 - u^2(1+v^2) - 2u^2v^2(1+v^2)}{(1+v^2)(1-u)}.$$

$$\left(\frac{v^2+2uv^4-u}{uv^2}\right) T(u,v) = \frac{u(v^2+2uv^4-u)}{(1+v^2)(1-u)}.$$

$$T(u,v) = \frac{u^2v^2}{(1+v^2)(1-u)}$$

$$\therefore \; T(u,v) = \frac{u^2}{(1-u)}\frac{v^2}{(1+v^2)} \qquad \text{............(30)}$$

Applying inverse double Elzaki transform of equation (30), we get an exact solution

$$u(x,t) = e^x \cos t .$$

CONCLUSIONS

PIDE's are used in modeling various phenomena in science, engineering and social sciences. The methods of Laplace , Elzaki and Double Elzaki transforms are successfully used to solve a general linear PIDE's. In Laplace and Elzaki transforms general linear PIDE's are solve by using convolution kernel. In double Elzaki transform by using an algebraic equation we solve general linear PIDE's. Finally we get exact solutions of such PIDE after a few steps of calculations.

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BIOGRAPHIES

Mrs. Gore(Jagtap) Jyotsana S.
She has completed M.Sc.(Maths) from Willingdon College -Sangli under Shivaji University Kolhapur. Her major fields of studies are various types of differential equations and fuzzy mathematics. Currently she is working as a lecturer in Mathematics department at MET's IOE BKC-Nashik.

Mr. Gore Shukracharya S.
He has completed M.E.(VLSI and Embedded Systems) from SITRC - Nashik under Pune University Pune. His major fields of studies are VLSI, Embedded Systems, Digital Signal Processing and ASIC's.
He has presented and published many papers in national and internationaljournals/Conferences. Currently he is working as a Assistant Professor in E&TC department at MCOE&RC-Nashik.

