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A Review on Complexity Results for Some Eigenvector Problems

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Abstract-*We consider the computation of eigenvectors x = (x1; : : : ; xn) over the integers, where each component xi satisfies jxij b for an integer b. We address various problems in this context, and analyze their computational complexity. We find that different problems are complete for the complexity classes NP, PNPk , FNP//OptP[O(log n)], FPNP, PNP, and NPNP. Applying the results, finding bounded solutions of a Diophantine equation v xT = 0 is shown to be intractable.*

Keywords: Mathematical programming, eigenvectors, problem complexity, combinatorial optimization

1 Introduction

Eigenvalues and eigenvectors have important applications in many areas, e.g. to problems in structural analysis, quantum chemistry, power system analysis, stability analysis, VLSI design, and geophysics [2]. The computation of eigenvalues and eigenvectors is thus an important problem, which has been investigated intensively in the past; see e.g. [3, 5, 11] and references therein.

In this paper, we address the complexity of computing distinguished elements out of the in general infinite set of eigenvectors for a given eigenvalue of a matrix M over the integers Z. In particular, we consider the computation of eigenvectors within a box of Zn, i.e., the set of vectors v = (v1; : : : ; vn) such that the absolute value jvij of each component vi is at most b; we call such vectors b-*bounded*. Observe that in programming languages, the range of integers is usually b-bounded for some constant b 1.As with the computation of eigenvectors, there is particular interest in computing shortest Eigenvectors, i.e., a non-zero eigenvector v such that its length kvk, which is understood in terms of the L2 (Euclidean) norm, is smallest. For this problem e.g. the algorithm of Hastad˚ et al. [6] for finding integer relationships between real vectors can be employed, which is closely related to the Lovasz-Lenstra-Lenstra (L3) algo- rithm [9]. Given linearly independent vectors v1; : : : ; vs 2 Zn, and k 0, the algorithm in [6] finds a vector x 2 Zn in polynomial time such that vi xT = 0 for all i = 1; : : : ; s or reports that no such vector of length 2k exists. The vector computed is not shortest, but usually shorter than a vector obtained by simple algorithm such as a standard Gaussian elimination. Furthermore, the algorithm does not return a b-bounded vector in general, and it is not clear whether the algorithm could be modified in this respect.

The main contributions of the present paper can be summarized as follows:

We give a precise characterization of the computational complexity of different problems in the context of computing b-bounded eigenvectors over Z. As we show, this problem is intractable in general. In particular, we show that computing a shortest b-boundedeigenvector is complete forFPNP and, if b is a constant, complete for the class FNP//OptP[O(log n)] introduced by Chen and Toda [1]. Few natural problems which are complete for this class are known so far.

By means of this complexity characterization, appropriate algorithm schemes for the solution of these problems emerge.

We provide several different problems, which can be used to establish similar hardness results for related problems.

2 Problem Statements

We assume tacitly that vectors and matrices are over the integers Z. We consider the following problems:

P Problem P1: Given an n n matrix M, an integer eigenvalue of M, a real number K, and a bound 1, does there exist a b-bounded non-zero eigenvector x for such that kxk K?1

This problem is the decision problem naturally associated with the problem of computing a shortest b-bounded eigenvector x. It is related to integer and quadratic programming problems (see [4]). We show pthat P1 is NP-complete, and hardness holds even if K = nb, i.e., deciding whether anyb-bounded eigenvector exists is NP-complete. Thus, the algorithm of Hastad˚ et al. [6] can not be modified to find a b-bounded nonzero integer relationship among vectors v1; : : : ; vs in polynomial time. As shown in Section 5, this holds even if s = 1, i.e., for a single vector.

Problem P2: Given an n n matrix M, an integer eigenvalue of M, and an integer b, compute a shortest eigenvector x among the b-bounded eigenvectors for intuitively, solving this problem requires computing the length kxk of a shortest b-boundedeigenvector, and generating an eigenvector of that norm. This problem is complete for FPNP in general, and for FNP/ /OptP[O(log n)] if b is fixed to any constant c 1

4 Complexity Results

For determining the complexity of problems P1–P5, we refer to variants of problems involving the class- sical satisfiability problem SAT. Let ' = fC1; : : : ; Cmg be a set of propositional clausesCi on variables
X. A truth assignment to X satisfies ', if each clause C 2 ' contains at least one literal (i.e., variable of negated variable) with value *true*. An assignment is *not-all-equal satisfying* (*nae-satisfying*) for ', if each clause in ' contains two literals that have different value according to; clearly, each nae-satisfying assignment for ' satisfies ' in the standard sense. Moreover, if is an nae-satisfying assignment, then also the complementary assignment, in which each variable has opposite truth value, is nae-satisfying.
Let ' = fC1; : : : ; Cmg be an instance of 3SAT, i.e., a set of propositional clauses Ci = i;1_ i;2_ i;3,

i= 1; : : : ; m on variables X = fx1; : : : ; xng. Then denote by '0 the set of the following clauses:

xj _ xj _ zj and xj _ xj _ :zj, for each j = 1 : : : ; n,

i;1 _ i;2 _ wi and 3;i _ x0 _ :wi, for each i = 1; : : : ; m

where x0, all zj, all xj , and all wi are fresh variables and i;j = x`, if i;j = x`,and i;j = x` if
i;j = :x`.

The following is easily verified. Let be an nae-satisfying assignment for '0. If (x0) = false, then

, restricted to X, satisfies '; if (x0) = true, then the complementary assignment , restricted to

X, satisfies '. On the other hand, if an assignment satisfies ', then is extendible to at least onenae-satisfying assignment of '0 in which x0 = false. Thus, we obtain the following.

Lemma 4.1 *Let ' be any 3SAT instance on variables* X. *Then, the nae-satisfying assignments of* '0*such that* (x0) = false, *correspond on the variables* X *1-1 to the satisfying assignments of '.*

As a consequence, deciding whether a SAT instance is satisfiable under nae-satisfaction(NAESAT) is NP-hard [4], even if all clauses have size 3 (NAE3SAT).

We now turn to Problem P1 from above, and obtain our first result.

5 Discussion and Conclusion

The results that we have derived in the previous section may be profitably used to derive similar complex- ity results for related problems. As an example, we consider the problem of finding integer relationships between numbers [6]. Given a real vector v, find a vector ofintegers x such that v xT = 0. If v is an in- teger vector, then the resulting Diophantine equation always has nonzero solutions. Finding a b-bounded nonzero x which satisfies this equation is intractable, however.

Theorem 5.1 Given an integer vector v = (v1; : : : ; vn) and b 0, deciding whether there is a nonzero-bounded vector x 2 Z such that v xT =0 *is* NP-*complete. Hardness holds for* b *fixed to any* c 1.

Proof. Obviously, a proper x can be guessed an checked in polynomial time. For the hardness part, we reduce problem P1 to this problem. Rewrite M xT = xT as M0 xT = 0, where M0 = M I (I
is the identity matrix). Let m = max jm0i;jj be the largest absolute value in M0. Define D = b n m + 1,
i;j
and let the vector v = (v1; : : : ; vn) be

$v_j = \sum D^{i-1} m_{i,j}$

By the same reduction, similar complexity results as for problems P2-P5 can be established for analogous problems on a single Diophantine equation $v \cdot x^T = 0$. In this paper, we have considered the computational difficulty of problems that arise in the context of computing bounded integer eigenvectors for a given integer matrix M and eigenvalue λ.

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