OBSERVATIONS ON THE BIQUADRATIC EQUATION WITH FIVE UNKNOWNS $2(x-y)(x^3+y^3)=(1+3k^2)(X^2-Y^2)w^2$

M.A.Gopalan[1]*, A.Kavitha[2], R.Kiruthika[3]

[1] Professor, Department of Mathematics, shrimati Indira Gandhi College, Tamilnadu, India

[2] Lecturer, Department of Mathematics, shrimati Indira Gandhi College, Tamilnadu, India

[3] Research Scholar, Department of Mathematics, shrimati Indira Gandhi College, Tamilnadu, India

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Abstract: We obtain infinitely many non-zero integer quintuples x, y, X, Y, w satisfying the biquadratic equation with five unknowns $2(x-y)(x^3+y^3)=(1+3k^2)(X^2-Y^2)w^2$ various interesting relations between the solutions and special numbers, octahedral numbers, centered polygonal & pyramidal numbers are exhibited.

Key Words: Bi-Quadratic equation with five unknowns, Integral solutions, polygonal number, Pyramidal numbers, Centered polygonal.

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NOTATIONS USED:

1. Polygonal number of rank 'n' with sides m

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

2. Stella octangular number of rank 'n'

$$SO_n=n(2n^2-1)$$

3. Pyramidal number of rank 'n' sides m

$$P_n^m=\frac{n(n+1)}{6}[(m-2)n+(5-m)]$$

4. Pronic number of rank 'n'

$$Pr_n=n(n+1)$$

5. Octahedral number of rank 'n'

$$OH_n=\frac{1}{3}[n(2n^2+1)]$$

1. INTRODUCTION:

Bi-quadratic Diophantine Equations, homogeneous and non- homogeneous, have aroused the interest of numerous

Mathematicians since ambiguity as can be seen from [1,2 ,17-19]. In the context one may refer [3-16] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of bi-quadratic equation in six unknowns represented by

$$2(x-y)(x^3+y^3)=(1+3k^2)(X^2-Y^2)w^2$$

A few interesting relations between the solutions and special polygonal numbers are presented.

2. METHOD OF ANALYSIS:

The Diophantine equation representing the biquadratic equation with five unknowns under consideration is

$$2(x-y)(x^3+y^3)=(1+3k^2)(X^2-Y^2)w^2 \qquad (1)$$

Introducing the linear transformations

$$x=u+v$$
$$y=u-v$$
$$X=2u+v \qquad (2)$$
$$Y=2u-v$$

in (1), it simplifies to

$$u^2+3v^2=(1+3k^2)w^2 \qquad (3)$$

The above equation (3) is solved through different methods and thus, one obtains distinct sets of integer solutions to (1)

2.1 set.1

Let $w=a^2+3b^2$ (4)

Substituting (4) in (3) and using the methods of factorization, define

$$(u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2 \qquad (5)$$

Equating real and imaginary parts, we have

$$u=a^2-3b^2-6abk$$
$$v=ka^2-3b^2k+2ab \qquad (6)$$

Substituting the values of u & v in (2), the non-zero distinct integral solutions of (1) are given by

$$x(a,b)=a^2+ka^2-3b^2-3b^2k+2ab-6abk$$
$$y(a,b)=a^2-ka^2-3b^2+3b^2k-2ab-6abk$$
$$X(a,b)=2a^2+ka^2-6b^2-3b^2k+2ab-12abk$$
$$Y(a,b)=2a^2-ka^2-6b^2+3b^2k-2ab-12abk$$
$$w(a,b)=a^2+3b^2 \qquad (7)$$

2.2 Properties:

A few interesting properties obtained as follows:

$(i) x(a,a+1)+y(a,a+1)+4(t_{4,a}-6kt_{3,a})\equiv 6(\mathrm{mod}12)$

$(ii) x(a,a+1)-y(a,a+1)+2k(2t_{4,a}+6a+3)-4p_{r_a}=0$

$(iii) x(a,2a^2+1)+y(a,2a^2+1)+36OH_a$
$+2(11t_{4,a}+12t_{4,a^2}+3)=0$

$(iv) x(a,2a^2-1)+w(a,2a^2-1)-2(t_{4,a}+SO_a)+$
$k(-13t_{4,a}+12t_{4,a^2}+6SO_a+3)=0$

$(v) x(a,2a-1)-w(a,2a-1)+2(12t_{4,a}-t_{6,a})+$
$k(11t_{4,a}+6t_{6,a}-12a+3)\equiv -6(\mathrm{mod}24)$

$(vi) X(a,3a-1)+Y(a,3a-1)+104t_{4,a}$
$+48kt_{5,a}\equiv -12(\mathrm{mod}72)$

$(vii) X(a,4a-3)-Y(a,4a-3)-$
$2k(-47t_{4,a}+72a-27)-4t_{10,a}=0$

2.3 Note:

In (5) replace $(1+i\sqrt{3}k)$ by $(-1+i\sqrt{3}k)$

$$\therefore (u+i\sqrt{3}v)=(-1+i\sqrt{3}k)(a+i\sqrt{3}b)^2 \quad (8)$$

Following the procedure presented in set. 1 a different solution is given by

$$\begin{aligned}
x(a,b)&=-a^2+ka^2+3b^2-3b^2k-2ab-6abk\\
y(a,b)&=-a^2-ka^2+3b^2+3b^2k+2ab-6abk\\
X(a,b)&=-2a^2+ka^2+6b^2-3b^2k-2ab-12abk\\
Y(a,b)&=-2a^2-ka^2+6b^2+3b^2k+2ab-12abk\\
w(a,b)&=a^2+3b^2
\end{aligned} \quad (9)$$

3.1 set.2

(3) can be written as

$$u^2+3v^2=(1+3k^2)w^2*1 \quad (10)$$

Write 1 as $1=\dfrac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$ (11)

Using (11) in (10) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2*\frac{(1+i\sqrt{3})}{2} \quad (12)$$

Equating real & imaginary parts & replacing a by 2a & b by 2b, we have

$$\begin{aligned}
u&=2a^2-6ka^2-6b^2+18b^2k\\
&\quad -12ab-12abk\\
v&=2a^2+2ka^2-6b^2-6b^2k\\
&\quad +4ab-12abk
\end{aligned} \quad (13)$$

Using (13) & (2) we get the integral solutions of (1) to be

$$x(a,b)=4a^2-4ka^2-12b^2+12b^2k-8ab-24abk$$
$$y(a,b)=-8ka^2+24b^2k-16ab$$
$$X(a,b)=6a^2-10ka^2-18b^2+30b^2k-20ab-36abk$$
$$Y(a,b)=2a^2-14ka^2-6b^2+42b^2k-28ab-12abk$$
$$w(a,b)=4a^2+12b^2$$
(14)

3.2 Properties:

A few interesting properties obtained as follows:

$(i).x(a,2a^2-1)+y(a,2a^2-1)+4(-13t_{4,a}+12t_{4,a^2}+6SO_a+3)+k(156t_{4,a}-144t_{4,a^2}+24SO_a-36)=0$

$(ii).x(a,a+1)-y(a,a+1)+8(t_{4,a}-P_{r_a})+4k(2t_{4,a}+6P_{r_a}+6a+3)\equiv -12(\text{mod}24)$

$(iii).x(a,2a^2+1)+w(a,2a^2+1)+8(-t_{4,a}-3OH_a)-2k(22t_{4,a}+24t_{4,a^2}-36OH_a+6)=0$

$(iv).x(a,a+1)-w(a,a+1)+8(3t_{4,a}+2t_{3,a})+4k(2t_{4,a}-12t_{3,a}+6a+3)\equiv -24(\text{mod}48)$

$(v).y(a^2,a+1)+w(a^2,a+1)+8k(t_{4,a^2}-3t_{4,a}-6a-3)+4(8P_a^5-t_{4,a^2}-3t_{4,a})\equiv 12(\text{mod}24)$

$(vi).y(a,2a-1)-w(a,2a-1)+4(13t_{4,a}+4t_{6,a})-4k(22t_{4,a}-24a+6)\equiv -12(\text{mod}48)$

$(vii).X(a,3a-2)+Y(a,3a-2)-8(-26t_{4,a}-6t_{8,a})+8k(78t_{4,a}-6t_{8,a}-72)\equiv -96(\text{mod}288)$

$(viii)X(a,7a-5)-Y(a,7a-5)-8(-73t_{4,a}+2t_{9,a})-4k(-146t_{4,a}-12t_{9,a}+210a-75)\equiv -300(\text{mod}840)$

3.3 Note:

In (12) replace $\frac{(1+i\sqrt{3})}{2}$ by $\frac{(-1+i\sqrt{3})}{2}$

$$\therefore (u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2*\frac{(-1+i\sqrt{3})}{2} \quad (15)$$

Following the procedure presented in set.2 a different solution is given by

$$x(a,b)=-8ka^2+24b^2k-16ab$$
$$y(a,b)=-4a^2-4ka^2+12b^2+12b^2k-8ab+24abk$$
$$X(a,b)=-2a^2-14ka^2+6b^2+42b^2k-28ab+12abk$$
$$Y(a,b)=-6a^2-10ka^2+18b^2+30b^2k-20ab+36abk$$
$$w(a,b)=4a^2+12b^2$$
(16)

Case 1:

$$u^2+3v^2=(1+3k^2)w^2*1$$

Instead of (11), write 1 as

$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49} \quad (17)$$

Using (17) in (10) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2*\frac{(1+i4\sqrt{3})}{7} \quad (18)$$

Equating real & imaginary parts & replacing a by 7a & b by 7b

$$\begin{aligned}u&=7a^2-84ka^2-21b^2+252b^2k-168ab-42abk\\ v&=28a^2+7ka^2-84b^2-21b^2k+14ab-168abk\end{aligned} \quad (19)$$

Using (19) & (2) we get the integral solutions of (1) to be

$$\begin{aligned}x(a,b)&=35a^2-77ka^2-105b^2+231b^2k\\&\quad-154ab-210abk\\ y(a,b)&=-21a^2-91ka^2+63b^2+\\&\quad 273b^2k-182ab+126abk\\ X(a,b)&=42a^2-161ka^2-126b^2\\&\quad+483b^2k-322ab-252abk\\ Y(a,b)&=-14a^2-175ka^2+42b^2\\&\quad+525b^2k-350ab+84abk\\ w(a,b)&=49a^2+147b^2\end{aligned} \quad (20)$$

3.4 Note

In (18) replace $\frac{(1+i4\sqrt{3})}{7}$ by

$\frac{(-1+i4\sqrt{3})}{7}$

$$\therefore (u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2*\frac{(-1+i4\sqrt{3})}{7} \quad (21)$$

Following the procedure presented in case 1, a different solution is given by

$$\begin{aligned}x(a,b)&=21a^2-91ka^2-63b^2\\&\quad+273b^2k-182ab-126abk\\ y(a,b)&=-35a^2-77ka^2+105b^2\\&\quad+231b^2k-154ab+210abk\\ X(a,b)&=14a^2-175ka^2-42b^2\\&\quad+525b^2k-350ab-84abk\\ Y(a,b)&=-42a^2-161ka^2+126b^2\\&\quad+483b^2k-322ab+252abk\\ w(a,b)&=49a^2+147b^2\end{aligned} \quad (22)$$

Case 2:

$$u^2+3v^2=(1+3k^2)w^2*1$$

Instead of (17), write 1 as

$$1=\frac{(11+i5\sqrt{3})(11-i5\sqrt{3})}{196} \quad (23)$$

Using (23) in (10) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2*\frac{(11+i5\sqrt{3})}{14} \quad (24)$$

Equating real & imaginary parts & replacing a by 14a & b by 14b

$$\begin{aligned}u&=154a^2-210ka^2-462b^2+\\&\quad 630b^2k-420ab-924abk\\ v&=70a^2+154ka^2-210b^2\\&\quad-462b^2k+308ab-420abk\end{aligned} \quad (25)$$

Using (25) & (2) we get the integral solutions of (1) to be

$$\begin{aligned} x(a,b) &= 224a^2 - 56ka^2 - 672b^2 \\ &\quad + 168b^2k - 112ab - 1344abk \\ y(a,b) &= 84a^2 - 364ka^2 - 252b^2 \\ &\quad + 1092b^2k - 728ab - 504abk \\ X(a,b) &= 378a^2 - 266ka^2 - 1134b^2 \\ &\quad + 798b^2k - 532ab - 2268abk \\ Y(a,b) &= 238a^2 - 574ka^2 - 714b^2 \\ &\quad + 1722b^2k - 1148ab - 1428abk \\ w(a,b) &= 196a^2 + 588b^2 \end{aligned} \quad (26)$$

3.5 Note

In (24) replace

$\frac{(11+i5\sqrt{3})}{14}$ by $\frac{(-11+i5\sqrt{3})}{14}$

$$\therefore (u+i\sqrt{3}v) = (1+i\sqrt{3}k)(a+i\sqrt{3}b)^2 * \frac{(-11+i5\sqrt{3})}{14} \quad (27)$$

Following the similar procedure as in case 2, the corresponding integer solutions of (1) are given by

$$\begin{aligned} x(a,b) &= -84a^2 - 364ka^2 + 252b^2 \\ &\quad + 1092b^2k - 728ab + 504abk \\ y(a,b) &= -224a^2 - 56ka^2 + 672b^2 \\ &\quad + 168b^2k - 112ab + 1344abk \\ X(a,b) &= -238a^2 - 574ka^2 + 714b^2 \\ &\quad + 1722b^2k - 1148ab + 1428abk \\ Y(a,b) &= -378a^2 - 266ka^2 + 1134b^2 \\ &\quad + 798b^2k - 532ab + 2268abk \\ w(a,b) &= 196a^2 + 588b^2 \end{aligned} \quad (28)$$

Case 3:

$$u^2 + 3v^2 = (1+3k^2)w^2 * 1$$

Instead of (23), write 1 as

$$1 = \frac{(13+i3\sqrt{3})(13-i3\sqrt{3})}{196} \quad (29)$$

Following the similar procedure as in case 2, the corresponding integer solutions of (1) are obtained as

$$\begin{aligned} x(a,b) &= 224a^2 + 56ka^2 - 672b^2 \\ &\quad - 168b^2k + 112ab - 1344abk \\ y(a,b) &= 140a^2 - 308ka^2 - 420b^2 \\ &\quad + 924b^2k - 616ab - 840abk \\ X(a,b) &= 406a^2 - 70ka^2 - 1218b^2 \\ &\quad + 210b^2k - 140ab - 2436abk \\ Y(a,b) &= 322a^2 - 434ka^2 - 966b^2 \\ &\quad + 1302b^2k - 868ab - 1932abk \\ w(a,b) &= 196a^2 + 588b^2 \end{aligned} \quad (30)$$

3.6 Note

In (29) replace $\frac{(13+i3\sqrt{3})}{14}$ by $\frac{(-13+i3\sqrt{3})}{14}$

$$\therefore (u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2 * \frac{(-13+i3\sqrt{3})}{14} \quad (31)$$

Following the similar procedure as in case 3, the corresponding integer solutions of (1) are as follows:

$$\begin{aligned} x(a,b) &= -140a^2 - 308ka^2 + 420b^2 \\ &\quad + 924b^2k - 616ab + 840abk \\ y(a,b) &= -224a^2 + 56ka^2 + 672b^2 \\ &\quad - 168b^2k + 112ab + 1344abk \\ X(a,b) &= -322a^2 - 434ka^2 + 966b^2 \\ &\quad + 1302b^2k - 868ab + 1932abk \\ Y(a,b) &= -406a^2 - 70ka^2 + 1218b^2 \\ &\quad + 210b^2k - 140ab + 2436abk \\ w(a,b) &= 196a^2 + 588b^2 \end{aligned} \quad (32)$$

Case 4:

$$u^2 + 3v^2 = (1+3k^2)w^2 * 1$$

Instead of (29), write 1 as

$$1 = \frac{(11+i4\sqrt{3})(11-i4\sqrt{3})}{169} \quad (33)$$

Using (33) in (10) and employing the method of factorization, define

$$(u+i\sqrt{3}v) = (1+i\sqrt{3}k)(a+i\sqrt{3}b)^2 * \frac{(11+i4\sqrt{3})}{13} \quad (34)$$

Equating real & imaginary parts & replacing a by 13a & b by 13b, we have

$$\begin{aligned} u &= 143a^2 - 156ka^2 - 429b^2 \\ &\quad + 468b^2k - 321ab - 858abk \\ v &= 52a^2 + 143ka^2 - 156b^2 \\ &\quad - 429b^2k + 286ab - 312abk \end{aligned} \quad (35)$$

Using (35) & (2) we get the integral solutions of (1) to be

$$\begin{aligned} x(a,b) &= 195a^2 - 13ka^2 - 585b^2 \\ &\quad - 39b^2k - 26ab - 1170abk \\ y(a,b) &= 91a^2 - 299ka^2 - 273b^2 \\ &\quad + 897b^2k - 598ab - 546abk \\ X(a,b) &= 338a^2 - 169ka^2 - 1014b^2 \\ &\quad + 507b^2k - 338ab - 2028abk \\ Y(a,b) &= 234a^2 - 455ka^2 - 702b^2 \\ &\quad + 1365b^2k - 910ab - 1404abk \\ w(a,b) &= 169a^2 + 507b^2 \end{aligned} \quad (36)$$

3.7 Note

In (34) replace $\frac{(11+i4\sqrt{3})}{13}$ by $\frac{(-11+i4\sqrt{3})}{13}$

$$\therefore (u+i\sqrt{3}v)=(1+i\sqrt{3}k)(a+i\sqrt{3}b)^2 * \frac{(-11+i4\sqrt{3})}{13} \quad (37)$$

Following the similar procedure as in case 3, the corresponding integer solutions of (1) are found to be

$$\begin{aligned} x(a,b) &= -91a^2 - 299ka^2 + 273b^2 \\ &\quad + 897b^2k - 598ab + 546abk \\ y(a,b) &= -195a^2 - 13ka^2 + 585b^2 \\ &\quad + 39b^2k - 26ab + 1170abk \\ X(a,b) &= -234a^2 - 455ka^2 + 702b^2 \\ &\quad + 1365b^2k - 910ab + 1404abk \\ Y(a,b) &= -338a^2 - 169ka^2 + 1014b^2 \\ &\quad + 507b^2k - 338ab + 2028abk \\ w(a,b) &= 169a^2 + 507b^2 \end{aligned} \qquad (38)$$

4.1 set.III

Rewrite (3), we get

$$u^2 - w^2 = 3(k^2w^2 - v^2) \qquad (39)$$

The above equation can be written in the form of ratio as

$$\frac{u+w}{kw+v} = \frac{3(kw-v)}{u-w} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (40)$$

(40) is equivalent to the system of double equations

$$\frac{u+w}{kw+v} = \frac{\alpha}{\beta}, \frac{3(kw-v)}{u-w} = \frac{\alpha}{\beta}$$

$$\begin{aligned} &\Rightarrow \beta u - \alpha v + (\beta - \alpha k)w = 0 \\ &\Rightarrow \alpha u + 3\beta v + (-3\beta k - \alpha)w = 0 \end{aligned} \qquad (41)$$

Solving the above equations by applying the method of cross multiplication, we have

$$\begin{aligned} u &= \alpha^2 - 3\beta^2 + 6\alpha\beta k \\ v &= -k\alpha^2 + 3\beta^2 k + 2\alpha\beta \end{aligned} \qquad (42)$$

Using the value of u & v in (2), we get the corresponding non-zero integer solutions to (1) to be

$$\begin{aligned} x(a,b) &= \alpha^2 - k\alpha^2 - 3\beta^2 \\ &\quad + 3\beta^2 k + 2\alpha\beta + 6\alpha\beta k \\ y(a,b) &= \alpha^2 + k\alpha^2 - 3\beta^2 \\ &\quad - 3\beta^2 k - 2\alpha\beta + 6\alpha\beta k \\ X(a,b) &= 2\alpha^2 - k\alpha^2 - 6\beta^2 \\ &\quad + 3\beta^2 k + 2\alpha\beta + 12\alpha\beta k \\ Y(a,b) &= 2\alpha^2 + k\alpha^2 - 6\beta^2 \\ &\quad - 3\beta^2 k - 2\alpha\beta + 12\alpha\beta k \\ w(a,b) &= \alpha^2 + 3\beta^2 \end{aligned} \qquad (43)$$

4.2 Properties:

A few interesting properties obtained as follows:

❖ $x(a,2a^2+1) + y(a,2a^2+1) - 2(-11t_{4,a} - 12t_{4,a^2} + 18kOH_a - 3) = 0$

❖ $x(a,2a^2-1) - y(a,2a^2-1) - 2k(-13t_{4,a} + 12t_{4,a^2} + 3) - 4SO_a = 0$

- $x(a,a+1)+w(a,a+1)-2(t_{4,a}+\Pr_a)$
  $-k(2t_{4,a}+6\Pr_a+6a+3)=0$

- $x(a^2,a+1)-w(a^2,a+1)$
  $-k(-t_{4,a^2}+3t_{4,a}+12P_a^5+6a+3)$
  $-2(-3t_{4,a}+2P_a^5)\equiv -6(\bmod 12)$

- $y(a,4a-3)+w(a,4a-3)-2(t_{4,a}-t_{10,a})$
  $-k(-47t_{4,a}+6t_{10,a}+72a-27)=0$

- $y(a,5a-3)-w(a,5a-3)$
  $-k(-74t_{4,a}+12t_{7,a}+90a-27)$
  $+2(75t_{4,a}+2t_{7,a})\equiv -54(\bmod 180)$

- $X(a,3a-2)+Y(a,3a-2)$
  $-8(-13t_{4,a}+3kt_{8,a})\equiv -48(\bmod 144)$

- $X(a,a+1)-Y(a,a+1)$
  $-2k(2t_{4,a}+6a+3)-8t_{3,a}=0$

In addition to (40), (39) may also be expressed in the form of ratios in three different cases that are presented below

Case 1:

$$\frac{u+w}{3(kw+v)}=\frac{kw-v}{u-w}=\frac{\alpha}{\beta},\beta\neq 0 \quad (44)$$

Case 2:

$$\frac{u+w}{3(kw-v)}=\frac{kw+v}{u-w}=\frac{\alpha}{\beta},\beta\neq 0 \quad (45)$$

Case 3:

$$\frac{u+w}{kw-v}=\frac{3(kw+v)}{u-w}=\frac{\alpha}{\beta},\beta\neq 0 \quad (46)$$

Solving each of the above system of equation by the following procedure as presented in set. 3, the corresponding integer solution to (1) are found to be as given below

4.3 Solution for case 1:

$$\begin{aligned}
x(a,b)&=3\alpha^2-3k\alpha^2-\beta^2\\
&\quad+\beta^2k+2\alpha\beta+6\alpha\beta k\\
y(a,b)&=3\alpha^2+3k\alpha^2-\beta^2\\
&\quad-\beta^2k-2\alpha\beta+6\alpha\beta k\\
X(a,b)&=6\alpha^2-3k\alpha^2-2\beta^2\\
&\quad+\beta^2k+2\alpha\beta+12\alpha\beta k\\
Y(a,b)&=6\alpha^2+3k\alpha^2-2\beta^2\\
&\quad-\beta^2k-2\alpha\beta+12\alpha\beta k\\
w(a,b)&=3\alpha^2+\beta^2
\end{aligned} \quad (47)$$

Solution for case 2:

$$\begin{aligned}
x(a,b)&=-3\alpha^2-3k\alpha^2+\\
&\quad\beta^2+\beta^2k+2\alpha\beta-6\alpha\beta k\\
y(a,b)&=-3\alpha^2+3k\alpha^2+\\
&\quad\beta^2-\beta^2k-2\alpha\beta-6\alpha\beta k\\
X(a,b)&=-6\alpha^2-3k\alpha^2+\\
&\quad 2\beta^2+\beta^2k+2\alpha\beta-12\alpha\beta k\\
Y(a,b)&=-6\alpha^2+3k\alpha^2+\\
&\quad 2\beta^2-\beta^2k-2\alpha\beta-12\alpha\beta k\\
w(a,b)&=-3\alpha^2-\beta^2
\end{aligned} \quad (48)$$

4.4 Solution for case

$$x(a,b) = -\alpha^2 - k\alpha^2 + 3\beta^2 + 3\beta^2 k + 2\alpha\beta - 6\alpha\beta k$$
$$y(a,b) = -\alpha^2 + k\alpha^2 + 3\beta^2 - 3\beta^2 k - 2\alpha\beta - 6\alpha\beta k$$
3: $$X(a,b) = -2\alpha^2 - k\alpha^2 + 6\beta^2 + 3\beta^2 k + 2\alpha\beta - 12\alpha\beta k \quad (49)$$
$$Y(a,b) = -2\alpha^2 + k\alpha^2 + 6\beta^2 - 3\beta^2 k - 2\alpha\beta - 12\alpha\beta k$$
$$w(a,b) = -\alpha^2 - 3\beta^2$$

5. Conclusion

In this paper, we have presented different choices of integer solutions to homogenous the biquadratic equation with five unknowns, $2(x-y)(x^3+y^3) = (1+3k^2)(X^2-Y^2)w^2$. It is worth mentioning here that in (2), the linear transformation for X & Y may also be considered as (1)X=uv+2, Y=uv-2 and (2) employing the above two forms of transformations for X, Y are obtained. To conclude, as bi-quadratic equations are rich in variety, one may consider other forms of bi-quadratic equations & search for corresponding properties.

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7. REFERENCES

[1] Carmicheal R.D, (1959),The theory of numbers and Diophantine analysis, Dover Publications, New York

[2] Dickson. L.E.,(1952), History of theory of numbers, Vol.2, Chelsia Publishing Co., New York

[3] Gopalan .M.A. and Anbuselvi.R,(2009), Integral solutions of binary quartic equation $x^3+y^3=(x-y)^4$, reflections des ERA-JMS, Vol.4 issue 3, pp271-280.

[4] Gopalan M.A and Janaki.G,(2009) Observation on$(x^2-y^2)4xy=z^4$, Acta ciencia Indica, vol.XXXVM, No.2,445

[5] Gopalan M.A,Vidhyalakshmi.S and Devibala.S,(2010), Ternary Quartic Diophantine Equation $2^{4n+3}(x^3-y^3)=z^4$, Impact journal of Science and Technology, Vol.4,No.57-60

[6] Gopalan M.A,Vijayasankar.A and Manju Somnath,(2010) Integral solutions of

$x^2 - y^2 = z^4$, Impact journal of Science and Technology, Vol.2(4),No.149-157

[7] Gopalan M.A and Shanmuganandham.P, (2010),On the Biquadratic equation $x^4 + y^4 + z^4 = 2w^4$ Impact journal of Science and Technology, Vol.4,No.4,111-115

[8] Gopalan M.A, Sangeetha.G,(2011) Integral solutions of ternary non-homogeneous biquadratic equations $x^4 + x^2 + y^2 - y = z^2 + z$, Acta ciencia indica, Vol.XXXVII M, No.4,799- 803

[9] Gopalan M.A,Vidhyalakshmi.S, Sumathi.G,(2012), On the ternary biquadruatic non- homogeneous equation $(2k + 1)(x^2 + y^2 + xy) = z^4$, Indian journal of Engineering, Vol.1, No.1

[10] Gopalan M.A, Vidhyalakshmi.S, Sumathi.G, (2012),Integral solutions of ternary biquadratic non-homogeneous equation$(\alpha + 1)(x^2 + y^2) + (2\alpha + 1)xy = z^4$, JARCE,Vol.6,No.2, 97-98,

[11] Gopalan M.A, Vidhyalakshmi.S, Sumathi.G,(2013), Integer solutions of ternary biquadratic non- homogeneous equation$(k + 1)(x^2 + y^2) - (2k + 1)xy = z^4$, Archimedes J.Math,3(1), 67-71.

[12] Gopalan M.A, Geetha.V,(2013), Integral solutions of ternary biquadratic equation $(x^2 + 13y^2) = z^4$, IJLRST, Vol.2, issue2, 59-61

[13] Gopalan M.A, Vidhyalakshmi.S, Sumathi.G, (2013),On the ternary biquadratic non-homogeneou equation $(x^2 + xy^3) = z^4$ Cayley J.Math, 2(2), 169-174

[14] Gopalan M.A, Vidhyalakshmi.S, Kavitha.A,(2013), Integral points on the biquadratic equation $(x + y + z)^3 = z^2(3xy - x^2 - y^2)$, IJMSEA,Vol.7,No.1,81-84

[15] Gopalan M.A, Vidhyalakshmi.S, Mallika.S,(2013), Integral solutions of $2(x^2 + y^2) + 3xy = (\alpha^2 + 7)^n z^4$, IJMIE,Vol.3, No.5, 408-414

[16] Gopalan M.A and Sivakami.B,(2013), Integral solutions of quadratic equation with four unknowns

$x^3 + y^3 + z^3 = 3xyz + 2(x + y)w^3$, Antarctica

J.math.,10(2) 151-159

[17] Mordell L.J, (1970), Diohantine Equations,

Academic Press, New York,

[18] Nigel, Smart.P, (1999),The Algorithmic

Resolutions of Diophantine Equations,

Cambridge University Press, London

[19] Telang S.G., (1996),Number Theory, Tata

Mcgraw Hill Publishing company, New Delhi

