On Fractional Operational Calculus pertaining to the product of H-functions

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Abstract-

The main aim of this paper is to obtain some results by applying two fractional integral operators of Oldham and Spanier [6] on the product H-function of one variable and H-function of several complex variables. The fractional integral formulae and H-functions taken here are in compact form and basic in nature. Some known and new results have been evaluated by taking different values of parameters. For sake of illustration, we mention here some special cases of our main results.

Key words and Phrases: Fractional integral operators, Multivariable H-function, Fox's H-function.

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1. INTRODUCTION

We start by introducing certain definitions:

The series representation of the Fox's H-function

$$H^{M,N}_{P,Q}\left[z\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]=\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,z^{\eta_G}, \tag{1}$$

Where $$\phi(\eta_G)=\frac{\prod\limits_{\substack{j=1\\j\neq g}}^{M}\Gamma(f_j-F_j\eta_G)\prod\limits_{j=1}^{N}\Gamma(1-e_j+E_j\eta_G)}{\prod\limits_{j=M+1}^{Q}\Gamma(1-f_j+F_j\eta_G)\prod\limits_{j=N+1}^{P}\Gamma(e_j-E_j\eta_G)}$$

and

$$\eta_G=\frac{f_g+G}{F_g}$$

One of the special case of H-function [5, p.11, eq.(1.7.8)] which is called Maitland's generalized hypergeometric function is as follows:

$$ {}_p\Psi_q\left[\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix};-z\right]=\sum_{r'=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+A_j r')(-z)^{r'}}{\prod_{j=1}^{q}\Gamma(b_j+B_j r')\;r'!} \tag{2}$$

The H-function of several complex variable [4] is defined as

$$H(z_1,\ldots,z_r)=H^{0,I:(H_1,I_1);\ldots;(H_r,I_r)}_{J,K:(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\theta'_j,\ldots,\theta^{(r)}_j)_{1,J}:(b'_j,\phi'_j)_{1,J_1};\ldots;(b^{(r)}_j,\phi^{(r)}_j)_{1,J_r}\\ (c_j;\psi'_j,\ldots,\psi^{(r)}_j)_{1,K}:(d'_j,\delta'_j)_{1,K_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,K_r}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int_{L_1}\cdots\int_{L_r}U_1(s_1)\ldots U_r(s_r)V(s_1,\ldots,s_r)z_1^{s_1}\ldots z_r^{s_r}\,ds_1\ldots ds_r, \tag{3}$$

where $\omega=\sqrt{-1}$ and

$$U_i(s_i)=\frac{\prod_{j=1}^{H_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{I_i}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_i)}{\prod_{j=1+H_i}^{K_i}\Gamma(1-d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1+I_i}^{J_i}\Gamma(b_j^{(i)}-\phi_j^{(i)}s_i)} \tag{4}$$

and

$$V(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{I}\Gamma\left(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i\right)}{\prod_{j=1+I}^{J}\Gamma\left(a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i\right)\prod_{j=1}^{K}\Gamma\left(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_i\right)}. \tag{5}$$

$i = 1,\ldots,r$

The convergence conditions and other details of the above function are given by Srivastava, Gupta and Goyal ([14], p.251, eq.(C.1), also see p.252-253, eq. (C.5 and C.6)).

In recent years, several authors Chen et al. [8], Soni et al. [9], Gaira et al. [2] have made significant contributions to the fractional integral operators pertaining to different functions and polynomials. Here we are trying to develop certain extensions of these results.

Oldham and Spanier [6] considered the fractional integral of a function f(z) of complex order ϕ as

$$ {}_gI_z^{\phi}[f(z)] = \begin{cases} \frac{1}{\Gamma\phi}\int_g^z (z-t)^{\phi-1} f(t)\,dt, \operatorname{Re}(\phi) > 0 \\ \frac{d^q}{dz^q}\, {}_gI_z^{\phi+q}[f(z)],\ \operatorname{Re}(\phi) \le 0\ , 0 < \operatorname{Re}(\phi)+q \le -1, \end{cases} \tag{6} $$

q = 1,2,3…

The special case of fractional integral operator ${}_gI_z^{\phi}$, when g = 0, will be denoted by I_z^{ϕ}. Thus, we write $I_z^{\phi} \equiv {}_0I_z^{\phi}$ and

$$ I_z^{\phi}[f(z)] = \begin{cases} \frac{1}{\Gamma\phi}\int_g^z (z-t)^{\phi-1} f(t)\,dt, \operatorname{Re}(\phi) > 0 \\ \frac{d^q}{dz^q}\, {}_gI_z^{\phi+q}[f(z)],\ \operatorname{Re}(\phi) \le 0\ , 0 < \operatorname{Re}(\phi)+q \le -1, q = 1,2,3,... \end{cases} \tag{7} $$

The above integral operator $I_z^{\phi}[f(z)]$ is called Riemann-Liouville fractional integral operator.

2. MAIN RESULTS

The results obtained by using fractional integral operators on certain functions are as follows:

(2.1):

$$ {}_gI_z^{\phi}\left\{ z^{\alpha} \prod_{i=1}^{t} (z+\beta_i)^{\tau_i} \prod_{j=1}^{k} H_{P_j,Q_j}^{M_j,N_j}\left[c_j z^{\gamma_j} \prod_{i=1}^{t} (z+\beta_i)^{\gamma_j^{(i)}} \right] \right. $$

$$ \left. .\, H\left[y_1 z^{\lambda_1} \prod_{i=1}^{t} (z+\beta_i)^{\lambda_1^{(i)}}, ..., y_r z^{\lambda_r} \prod_{i=1}^{t} (z+\beta_i)^{\lambda_r^{(i)}} \right] \right\} $$

$$ = z^{\alpha} (\beta_1)^{\tau_1} ... (\beta_t)^{\tau_t} \sum_{\ell_1=0}^{\infty} ... \sum_{\ell_t=0}^{\infty} \sum_{T=0}^{\infty} \sum_{G_1=0}^{\infty} \sum_{g_1=0}^{M_1} ... \sum_{G_k=0}^{\infty} \sum_{g_k=0}^{M_k} $$

$$.\, \frac{(-1)^{G_1} \phi(\eta_{G_1})(c_1)^{\eta_{G_1}}}{G_1!\, F_{G_1}} ... \frac{(-1)^{G_k} \phi(\eta_{G_k})(c_k)^{\eta_{G_k}}}{G_k!\, F_{G_k}} $$

$$\cdot\frac{(\beta_1)^{(\gamma_1^1\eta_{G_1}+\ldots+\gamma_k^1\eta_{G_k}-\ell_1)}}{\ell_1\,!}\ldots\frac{(\beta_t)^{(\gamma_1^{(t)}\eta_{G_1}+\ldots+\gamma_k^{(t)}\eta_{G_k}-\ell_t)}}{\ell_t\,!}$$

$$\cdot\frac{(-1)^T\,(z-g)^{T+\phi}}{\Gamma\phi(T+\phi)\;T\,!}\,z^{[\gamma_1\eta_{G_1}+\ldots+\gamma_k\eta_{G_k}+\ell_1+\ldots+\ell_t-T]}$$

$$\cdot H^{0,\,I+(t+1)\qquad :(H_1,I_1);\ldots;(H_r,I_r)}_{J+(t+1),K+(t+1):(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{matrix} y_1(\beta_1)^{\lambda_1^{(1)}}\ldots(\beta_t)^{\lambda_1^{(t)}}z^{\lambda_1}\\ \vdots\\ y_r(\beta_1)^{\lambda_r^{(1)}}\ldots(\beta_t)^{\lambda_r^{(t)}}z^{\lambda_r}\end{matrix}\right.$$

$$\left|\begin{matrix}(-\alpha-\gamma_1\eta_{G_1}-\ldots-\gamma_k\eta_{G_k}-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),\,(-\tau_1-\gamma_1^{(1)}\eta_{G_1}-\ldots-\gamma_k^{(1)}\eta_{G_k};\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\\ (T-\alpha-\gamma_1\eta_{G_1}-\ldots-\gamma_k\eta_{G_k}-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),(\ell_1-\tau_1-\gamma_1^{(1)}\eta_{G_1}-\ldots-\gamma_k^{(1)}\eta_{G_k};\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}(-\tau_t-\gamma_1^{(t)}\eta_{G_1}-\ldots-\gamma_k^{(t)}\eta_{G_k};\lambda_1^{(t)},\ldots,\lambda_r^{(1)}),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r}\\ (\ell_t-\tau_t-\gamma_1^{(t)}\eta_{G_1}-\ldots-\gamma_k^{(t)}\eta_{G_k};\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r}\end{matrix}\right]. \tag{8}$$

Also, by using the following result in (8)

$$\sum_{T=0}^{\infty}\frac{(-1)^T}{\Gamma\phi(T+\phi)T\,!}H^{0,I\quad :(H_1,I_1);\ldots;(H_r,I_r)}_{J,\,K+1:(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r}\\ (T-k;\gamma_1,\ldots,\gamma_r),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r}\end{matrix}\right.\right]$$

$$=H^{0,I:(H_1,I_1);\ldots;(H_r,I_r)}_{J,K+1:(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r}\\ (-\phi-k;\gamma_1,\ldots,\gamma_r),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r}\end{matrix}\right.\right] \tag{9}$$

We obtain

$$ {}_g I_z^{\phi}\left\{ z^{\alpha} \prod_{i=1}^{t} (z+\beta_i)^{\tau_i} \prod_{j=1}^{k} H_{P_j,Q_j}^{M_j,N_j}\left[c_j z^{\gamma_j} \prod_{i=1}^{t} (z+\beta_i)^{\gamma_j^{(i)}} \right]\right. $$

$$.\, H\left[y_1 z^{\lambda_1} \prod_{i=1}^{t} (z+\beta_i)^{\lambda_1^{(i)}}, \ldots, y_r z^{\lambda_r} \prod_{i=1}^{t} (z+\beta_i)^{\lambda_r^{(i)}} \right]\Bigg\} $$

$$ = z^{\alpha+\phi} (\beta_1)^{\tau_1} \ldots (\beta_t)^{\tau_t} \sum_{\ell_1=0}^{\infty} \ldots \sum_{\ell_t=0}^{\infty} \sum_{G_1=0}^{\infty} \sum_{g_1=0}^{M_1} \ldots \sum_{G_k=0}^{\infty} \sum_{g_k=0}^{M_k} $$

$$.\, \frac{(-1)^{G_1} \phi(\eta_{G_1})(c_1)^{\eta_{G_1}}}{G_1!\, F_{G_1}} \ldots \frac{(-1)^{G_k} \phi(\eta_{G_k})(c_k)^{\eta_{G_k}}}{G_k!\, F_{G_k}} $$

$$.\, \frac{(\beta_1)^{(\gamma_1^1 \eta_{G_1} + \ldots + \gamma_k^1 \eta_{G_k} - \ell_1)}}{\ell_1!} \ldots \frac{(\beta_t)^{(\gamma_1^{(t)} \eta_{G_1} + \ldots + \gamma_k^{(t)} \eta_{G_k} - \ell_t)}}{\ell_t!} z^{[\gamma_1 \eta_{G_1} + \ldots + \gamma_k \eta_{G_k} + \ell_1 + \ldots + \ell_t]} $$

$$.\, H^{0,\, I+(t+1) \;:(H_1,I_1);\ldots;(H_r,I_r)}_{J+(t+1),K+(t+1):(J_1,K_1);\ldots;(J_r,K_r)} \left[\begin{array}{c} y_1 (\beta_1)^{\lambda_1^{(1)}} \ldots (\beta_t)^{\lambda_1^{(t)}} z^{\lambda_1} \\ \vdots \\ y_r (\beta_1)^{\lambda_r^{(1)}} \ldots (\beta_t)^{\lambda_r^{(t)}} z^{\lambda_r} \end{array} \right. $$

$$ \left| \begin{array}{l} (-\alpha-\gamma_1\eta_{G_1}-\ldots-\gamma_k\eta_{G_k}-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),(-\tau_1-\gamma_1^{(1)}\eta_{G_1}-\ldots-\gamma_k^{(1)}\eta_{G_k};\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots, \\ (-\phi-\alpha-\gamma_1\eta_{G_1}-\ldots-\gamma_k\eta_{G_k}-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),(\ell_1-\tau_1-\gamma_1^{(1)}\eta_{G_1}-\ldots-\gamma_k^{(1)}\eta_{G_k};\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots, \end{array} \right. $$

$$ \left. \begin{array}{l} (-\tau_t-\gamma_1^{(t)}\eta_{G_1}-\ldots-\gamma_k^{(t)}\eta_{G_k};\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r} \\ (\ell_t-\tau_1-\gamma_1^{(t)}\eta_{G_1}-\ldots-\gamma_k^{(t)}\eta_{G_k};\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r} \end{array} \right], \quad (10) $$

provided that

(i) Re(ϕ) > 0 ; the quantities $\gamma_1, \gamma_1^{(1)}, \ldots, \gamma_k^{(1)}, \ldots, \gamma_k, \gamma_1^{(t)}, \ldots, \gamma_k^{(t)}, \lambda_1, \lambda_1^{(1)}, \ldots, \lambda_1^{(t)}, \quad \lambda_r, \lambda_r^{(1)}, \ldots, \lambda_t^{(t)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning),

(ii) $$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}\lambda_i \min_{1\le j\le M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0$$

(2.2):

$$I_z^{\theta,\phi}\left\{z^{\alpha}\prod_{i=1}^{t}(z+\beta_i)^{\tau_i}\prod_{j=1}^{k}H_{P_j,Q_j}^{M_j,N_j}\left[c_j z^{\gamma_j}\prod_{i=1}^{t}(z+\beta_i)^{\gamma_j^{(i)}}\right]\right.$$

$$.\left.H\left[y_1 z^{\lambda_1}\prod_{i=1}^{t}(z+\beta_i)^{\lambda_1^{(i)}},\ldots,y_r z^{\lambda_r}\prod_{i=1}^{t}(z+\beta_i)^{\lambda_r^{(i)}}\right]\right\}$$

$$=z^{\alpha}(\beta_1)^{\tau_1}\ldots(\beta_t)^{\tau_t}\sum_{G_1=0}^{\infty}\sum_{g_1=1}^{M_1}\ldots\sum_{G_k=0}^{\infty}\sum_{g_k=1}^{M_k}$$

$$.\frac{(-1)^{G_1}\phi(\eta_{G_1})(c_1)^{\eta_{G_1}}}{G_1!\,F_{G_1}}\ldots\frac{(-1)^{G_k}\phi(\eta_{G_k})(c_k)^{\eta_{G_k}}}{G_k!\,F_{G_k}}$$

$$.\sum_{\ell_1=0}^{\infty}\ldots\sum_{\ell_t=0}^{\infty}\frac{(\beta_1)^{\gamma_1^1\eta_{G_1}+\ldots+\gamma_k^1\eta_{G_k}-\ell_1}}{\ell_1!}\ldots\frac{(\beta_t)^{\gamma_1^{(t)}\eta_{G_1}+\ldots+\gamma_k^{(t)}\eta_{G_k}-\ell_t}}{\ell_t!}z^{\gamma_1\eta_{G_1}+\ldots+\gamma_k\eta_{G_k}+\ell_1+\ldots+\ell_t}$$

$$.H_{J+(t+1),K+(t+1):(J_1,K_1);\ldots;(J_r,K_r)}^{0,\,I+(t+1):(H_1,I_1);\ldots;(H_r,I_r)}\left[\begin{matrix}y_1(\beta_1)^{\lambda_1^{(1)}}\ldots(\beta_t)^{\lambda_1^{(t)}}z^{\lambda_1}\\ \vdots\\ y_r(\beta_1)^{\lambda_r^{(1)}}\ldots(\beta_t)^{\lambda_r^{(t)}}z^{\lambda_r}\end{matrix}\right.$$

$$\left|\begin{matrix}(1-\theta-\alpha-\gamma_1\eta_{G_1}-\ldots-\gamma_k\eta_{G_k}-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),(-\tau_1-\gamma_1^{(1)}\eta_{G_1}-\ldots-\gamma_k^{(1)}\eta_{G_k};\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\\ (1-\theta-\alpha-\phi-\gamma_1\eta_{G_1}-\ldots-\gamma_k\eta_{G_k}-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),(\ell_1-\tau_1-\gamma_1^{(1)}\eta_{G_1}-\ldots-\gamma_k^{(1)}\eta_{G_k};\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}(-\tau_t-\gamma_1^{(t)}\eta_{G_1}-\ldots-\gamma_k^{(t)}\eta_{G_k};\lambda_1^{(t)},\ldots,\lambda_r^{(1)}),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r}\\ (\ell_t-\tau_t-\gamma_1^{(t)}\eta_{G_1}-\ldots-\gamma_k^{(t)}\eta_{G_k};\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r}\end{matrix}\right],$$

(11)

provided that

(i) $\theta > 0$; the quantities $\gamma_1, \gamma_1^{(1)}, ..., \gamma_k^{(1)}, ..., \gamma_k, \gamma_1^{(t)}, ..., \gamma_k^{(t)}, \lambda_1, \lambda_1^{(1)}, ..., \lambda_1^{(t)}$, $\lambda_r, \lambda_r^{(1)}, ..., \lambda_t^{(t)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning),

(ii) $\mathrm{Re}(\alpha) + \sum_{i=1}^{r} \lambda_i \min_{1 \le j \le M^{(i)}} [\mathrm{Re}(d_j^{(i)} / \delta_j^{(i)})] + \theta > 0$.

Proof:

In order to prove (8), we first express the H-function of one variable in series form given by (1) and multivariable H-function in terms of Mellin-Barnes type of contour integrals and interchanging the order of summations, integration and taking the fractional integral operator inside, which is permissible under the stated conditions. Now, using binomial expansion along with the use of the known formula (6) and interpreting the multiple Mellin-Barnes contour integral so obtained in terms of H-function, we easily arrive at the desired result (8).

Also, using the same method adopted in the proof of the result (8) and making use of the formula [2, eq. (2.10)]

$$I_z^{\theta,\phi}[z^{\mu}] = \frac{\Gamma(\mu+\theta)}{\Gamma(\mu+\theta+\phi)} z^{\mu}, \ \mathrm{Re}\,(\mu) > -\theta$$

we can prove the result (11).

3. PARTICULAR CASES

3.1.

Replace $H_{P_j,Q_j}^{M_j,N_j}\left[c_j z^{\gamma_j} \prod_{i=1}^{t} (z+\beta_i)^{\gamma_j^{(i)}}\right]$ by ${}_{p_j}\psi_{q_j}\left[-c_j z^{\gamma_j} \prod_{i=1}^{t} (z+\beta_i)^{\gamma_j^{(i)}}\right]$

in result (10) and using (2), we get

$$_g I_z^{\phi}\left\{z^{\alpha} \prod_{i=1}^{t} (z+\beta_i)^{\tau_i} \prod_{j=1}^{k} {}_{p_j}\psi_{a_j}\left[-c_j z^{\gamma_j} \prod_{i=1}^{t} (z+\beta_i)^{\gamma_j^{(i)}}\right]\right.$$

$$.\ H\left[\left. y_1 z^{\lambda_1} \prod_{i=1}^{t} (z+\beta_i)^{\lambda_1^{(i)}}, ..., y_r z^{\lambda_r} \prod_{i=1}^{t} (z+\beta_i)^{\lambda_r^{(i)}}\right]\right\}$$

$$= z^{\alpha+\phi}(\beta_1)^{\tau_1}\ldots(\beta_t)^{\tau_t}\sum_{s_1=0}^{\infty}\ldots\sum_{s_k=0}^{\infty}\frac{\prod_{j=1}^{p_1}\Gamma(a_j+A_js_1)(-1)^{s_1}(c_1)^{s_1}}{\prod_{j=1}^{q_1}\Gamma(b_j+B_js_1)\,s_1!}\ldots$$

$$.\frac{\prod_{j=1}^{p_k}\Gamma(a_j+A_js_k)(-1)^{s_k}(c_k)^{s_k}}{\prod_{j=1}^{q_k}\Gamma(b_j+B_js_k)\,s_k!}\sum_{\ell_1=0}^{\infty}\ldots\sum_{\ell_t=0}^{\infty}\frac{(\beta_1)^{\gamma_1^1 s_1+\ldots+\gamma_k^1 s_k-\ell_1}}{\ell_1!}\ldots$$

$$.\frac{(\beta_t)^{\gamma_1^{(t)}s_1+\ldots+\gamma_k^{(t)}s_k-\ell_t}}{\ell_t!}\,z^{\gamma_1 s_1+\ldots+\gamma_k s_k+\ell_1+\ldots+\ell_t+\phi}$$

$$.H^{0,\,I+(t+1)\,:(H_1,I_1);\ldots;(H_r,I_r)}_{J+(t+1),K+(t+1):(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{matrix} y_1(\beta_1)^{\lambda_1^{(1)}}\ldots(\beta_t)^{\lambda_1^{(t)}}z^{\lambda_1} \\ \vdots \\ y_r(\beta_1)^{\lambda_r^{(1)}}\ldots(\beta_t)^{\lambda_r^{(t)}}z^{\lambda_r}\end{matrix}\right.$$

$$\left|\begin{matrix}(-\alpha-\gamma_1s_1-\ldots-\gamma_ks_k-\ell_1-\ldots-\ell_t;\lambda_1,\ldots,\lambda_r),(-\tau_1-\gamma_1^{(1)}s_1-\ldots-\gamma_k^{(1)}s_k;\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots, \\ (-\alpha-\gamma_1s_1-\ldots-\gamma_ks_k-\ell_1-\ldots-\ell_t-\phi;\lambda_1,\ldots,\lambda_r),(\ell_1-\tau_1-\gamma_1^{(1)}s_1-\ldots-\gamma_k^{(1)}s_k;\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}(-\tau_t-\gamma_1^{(t)}s_1-\ldots-\gamma_k^{(t)}s_k;\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r} \\ (\ell_t-\tau_t-\gamma_1^{(t)}s_1-\ldots-\gamma_k^{(t)}s_k;\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r}\end{matrix}\right],$$

(12)

the conditions for this result are same as the conditions for result (10).

3.2.

Replace $H^{M_j,N_j}_{P_j,Q_j}\left[c_j z^{\gamma_j}\prod_{i=1}^{t}(z+\beta_i)^{\gamma_j^{(i)}}\right]$ by ${}_{p_j}\Psi_{q_j}\left[-c_j z^{\gamma_j}\prod_{i=1}^{t}(z+\beta_i)^{\gamma_j^{(i)}}\right]$

in result (11) and using (2), we get

$$I_z^{\theta,\phi}\left\{z^{\alpha}\prod_{i=1}^{t}(z+\beta_i)^{\tau_i}\prod_{j=1}^{k}{}_{p_j}\psi_{q_j}\left[-c_j z^{\gamma_j}\prod_{i=1}^{t}(z+\beta_i)^{\gamma_j^{(i)}}\right]\right.$$

$$\left.\cdot H\left[y_1 z^{\lambda_1}\prod_{i=1}^{t}(z+\beta_i)^{\lambda_1^{(i)}},\ldots,y_r z^{\lambda_r}\prod_{i=1}^{t}(z+\beta_i)^{\lambda_r^{(i)}}\right]\right\}$$

$$= z^{\alpha}(\beta_1)^{\tau_1}\ldots(\beta_t)^{\tau_t}\sum_{s_1=0}^{\infty}\ldots\sum_{s_k=0}^{\infty}\frac{\prod_{j=1}^{p_1}\Gamma(a_j+A_j s_1)(-1)^{s_1}(c_1)^{s_1}}{\prod_{j=1}^{q_1}\Gamma(b_j+B_j s_1)\, s_1!}\ldots$$

$$\cdot\frac{\prod_{j=1}^{p_k}\Gamma(a_j+A_j s_k)(-1)^{s_k}(c_k)^{s_k}}{\prod_{j=1}^{q_k}\Gamma(b_j+B_j s_k)\, s_k!}\sum_{\ell_1=0}^{\infty}\ldots\sum_{\ell_t=0}^{\infty}\frac{(\beta_1)^{\gamma_1^1 s_1+\ldots+\gamma_k^1 s_k-\ell_1}}{\ell_1!}\ldots$$

$$\cdot\frac{(\beta_t)^{\gamma_1^{(t)}s_1+\ldots+\gamma_k^{(t)}s_k-\ell_t}}{\ell_t!}\, z^{\gamma_1 s_1+\ldots+\gamma_k s_k+\ell_1+\ldots+\ell_t}$$

$$\cdot H^{0,\,I+(t+1):(H_1,I_1);\ldots;(H_r,I_r)}_{J+(t+1),K+(t+1):(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{matrix} y_1(\beta_1)^{\lambda_1^{(1)}}\ldots(\beta_t)^{\lambda_1^{(t)}}z^{\lambda_1}\\ \vdots \\ y_r(\beta_1)^{\lambda_r^{(1)}}\ldots(\beta_t)^{\lambda_r^{(t)}}z^{\lambda_r}\end{matrix}\right.$$

$$\left|\begin{matrix}(1-\alpha-\gamma_1 s_1-\ldots-\gamma_k s_k-\ell_1-\ldots-\ell_t-\theta;\lambda_1,\ldots,\lambda_r),(-\tau_1-\gamma_1^{(1)}s_1-\ldots-\gamma_k^{(1)}s_k;\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\\ (1-\alpha-\gamma_1 s_1-\ldots-\gamma_k s_k-\ell_1-\ldots-\ell_t-\theta-\phi;\lambda_1,\ldots,\lambda_r),(\ell_1-\tau_1-\gamma_1^{(1)}s_1-\ldots-\gamma_k^{(1)}s_k;\lambda_1^{(1)},\ldots,\lambda_r^{(1)}),\ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}(-\tau_t-\gamma_1^{(t)}s_1-\ldots-\gamma_k^{(t)}s_k;\lambda_1^{(t)},\ldots,\lambda_r^{(1)}),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,J}:(b_j',\phi_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,J_r}\\ (\ell_t-\tau_1-\gamma_1^{(t)}s_1-\ldots-\gamma_k^{(t)}s_k;\lambda_1^{(t)},\ldots,\lambda_r^{(t)}),(c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,K}:(d_j',\delta_j')_{1,K_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,K_r}\end{matrix}\right].$$

(13)

The other conditions for this result are same as the conditions for result (11).

3.3.

Particularly, when we substitute t = 1 and $c_j = 0$ (j = 1,...,k) in our integral (10), we arrive on the result obtained by Srivastava et al. [11].

3.4.

If we put t = 1 = k and giving suitable values to the parameters in our integral formula (8), we can obtain another result obtained by Gupta et al. [3].

4. CONCLUSION

In this paper, we have applied the operators of Oldham and Spanier on the product of functions of general nature in one or more variables. By virtue of this we have been able to give certain formulae for fractional integral involving H-functions in compact form which are basic in nature .It is believed that the formulae truth to be generalization of many results scattered hitherto in the literature .

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