A Review of Graph Theory and Its Applications Across Various Disciplines

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ABSTRACT

This review paper offers a comprehensive analysis how graph theory may be applied to solve problems in a variety of fields, including computer science, environmental geography, power systems, and medical applications. The main goals of power systems evaluation are reliability improvement and vulnerability assessment through the use of game theory and graph theory models. In computer science, novel approaches to network optimisation and cybersecurity reinforcement are demonstrated by IoT-based machine learning, graph neural networks, and link prediction. Studies in medicine demonstrate how important graph theory is to understanding the relationships between brain shape and function, identifying cancer, and understanding how brain networks related to pain are organised. Graph theory is important for landscape connectivity, genetic diversity, and ecological network design in environmental applications because it sheds light on ecological structures, the effects of watershed development, and spatial patterns. Overall, the review highlights graph theory's wide significance and prospective research directions across disciplinary boundaries.

1. INTRODUCTION:

Graph theory is a significant field in Discrete Mathematics and serves as an important area of mathematical coordination. The modelling of objects as vertices and the representation of relationships as edges, offering a powerful approach to modelling complex situations **(Majeed & Rauf, 2020).** Graphs are critical for explaining complicated structures, from sports competitions like football tournaments to the complexities of social networks and the Internet as a whole **(Miz et al., 2019).**

Graph theory is a versatile technique that can be used to map social relationships between individuals or to define sports teams and their results **(Duque, ML Martins & Manuel Clemente, 2016).** In essence, it provides a mathematical foundation and serves as a strong analytical framework for understanding and modelling complex, interconnected systems across multiple domains **(Duindam et al., 2009).**

In the field of computer science research, graph theory has numerous applications in areas such as data mining, image capture, segmentation, networking, and clustering **(Ahmed, 2012).** It is useful for practical tasks including mail delivery route optimisation, troubleshooting and correcting network issues, and topologically based strategic local area network (LAN) planning **(Makeri, 2019).**

Complex problems such as understanding the dynamics of bilateral institutions and negotiating the complexities of the connections between employers and job seekers are addressed by scientific research into graph theory **(Ramalingam et al., 2008).**

Graph theory plays a key role in directing marketing tactics and advancing both business and agriculture **(King et al., 2010).** Following the principles of graph theory, individual points are called vertices, and the edges—the connections between them—specify the relationships between them **(Wilson, 1979).** This methodical abstraction enables a comprehensive analysis of interrelated components, offering a rigorous and useful analytical tool for strategic decision-making in the fields of business and agricultural growth **(Mollinga & Gondhalekar, 2014).**

"Fig. 1." is a simple graph. Here we can see in the graph G consists of a no empty finite set. $V(G)$ of segments called vertices (nodes) and a finite set $E(G)$of distinct unordered pairs of distinct elements of $V(G)$ called edge. A graph is a join $G = (V, E)$of sets of fulfilment $E \subseteq [v]2$; We will always take $V \cap E = \emptyset$ to avoid the ambiguity of notation. The Elements of v are the vertices or nodes of the graph. The elements of E are the edge. Here, the graph on $V = A, B, C, D, E$ and $E = D, A, D, E, D, C, E, C, A, E, A, B, B, E$

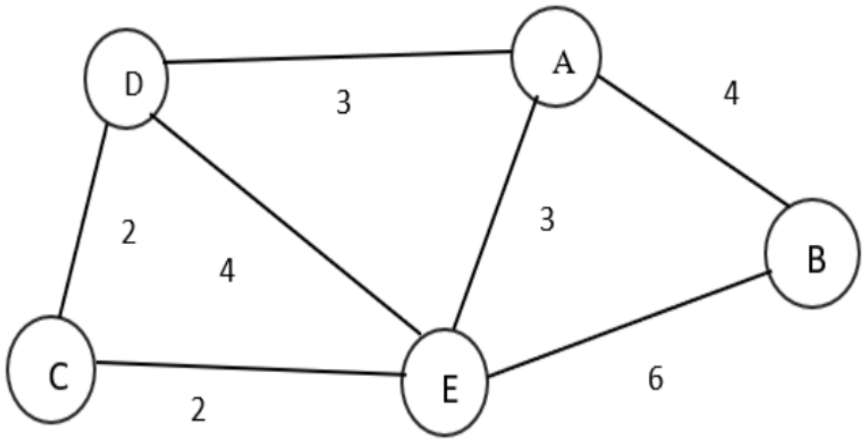

Fig 1: Example of Graph

Definitions:

Consider a finite group G. The graph, known as the **Commuting Graph of G**, is formed by using the elements of G as vertices. In this graph, two vertices x and y are connected if their product xy is equal to yx.

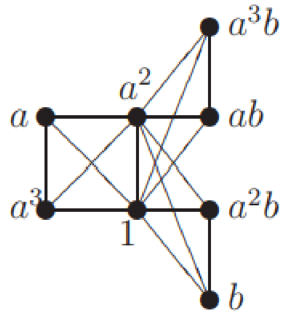

Fig 2: Commuting graph of $\boldsymbol{D_8}$ or $\boldsymbol{Q_8}$

The **directed power graph** of G is the directed graph with vertex set G, with an arc $x \rightarrow y$ if $y = xm$ for some integer m. The power graph of G is the graph obtained by ignoring directions and double arcs; in other words, x is joined to y if one of x and y is a power of the other. It is clearly a spanning subgraph of the commuting graph. A finite group G's **generating graph** has a vertex set called G, where x and y are joined if and only if $hx, yi = G$. The generating graph is the null graph if there are more generators in G than there are minimums. If G is cyclic, then loops exist in its generating graph. The set of prime divisors of the order of a finite group G is the vertex set of the **Gruenberg-Kegel graph, or prime graph**; vertices p and q are related by an edge if and only if G includes an element of order pq. Assume that G is a finite group that is neither trivial nor prime-order cyclic. If $H_1 \cap H_2$ not equal to {1}then two vertices, H1 and H2, are near in the graph that is the **intersection graph of G**. The graph's vertices are the non-trivial proper subgroups of G.

Graph Theory Formulas:

Degree of a Vertex (Undirected Graph):

- $Degree(v) = Numberofedgesincidenttov$

Degree of a Vertex (Directed Graph):

- $In - Degree(v) = Number\ of\ edges\ entering\ v$
- $Out - Degree(v) = Number\ of\ edges\ leaving\ v$

Adjacency Matrix (Undirected Graph):

- $A_{ij} = \begin{cases} 1 & if\ there\ is\ an\ edge\ between\ vi\ and\ vj \\ 0 & Otherwise \end{cases}$

Adjacency Matrix (Directed Graph):

- $A_{ij} = \begin{cases} 1 & if\ there\ is\ a\ directed\ edge\ \ from\ vi\ and\ vj \\ 0 & Otherwise \end{cases}$

Engineering Application Example: Network Flow Analysis in Electrical Engineering

In network flow analysis, graph theory is applied to model and optimize the flow of electrical currents in a network. Consider an electrical network represented by a directed graph where nodes represent junctions or components, and directed edges represent electrical connections. The current flowing through each edge can be represented by the following mathematical formula:

$$I_{ij} = \frac{V_i - V_j}{R_{ij}}$$

Where:

- $I_{ij}\ is\ the\ current\ flowing\ from\ node\ i\ to\ node\ j$.
- V_i and V_j are the voltages at nodes i and j respectively.
- R_{ij} is the resistance of the edge connecting nodes i and j.

The goal is to optimize the network flow by adjusting voltages or resistances based on the constraints and objectives of the electrical system. Graph theory provides a systematic approach to model and analyze such network flow problems in engineering applications.

Graphs and networks are essential tools for decision assistance because they apply exact mathematical solutions based on predetermined principles and use graphical models to show the complexity of choice problems **(Varbanov, Friedler & Klemes, 2017).** The theory of graphs and networks, which is based on operational research, is a vital component that improves methodical decision-making in organisational contexts **(Gibbons, 1997).**

Adoption of graph theory-based information systems becomes strategically necessary to maximise operational efficiency and technical competitiveness inside organisational frameworks **(Grover et al., 2018).**

Examining "Graph Theory and Its Application in Different Fields" is essential due to its broad impact on many academic fields, which enables the development of complex algorithms and the depiction of real-world situations. Understanding graph theory is essential for professionals and researchers who want to solve problems creatively, advance science and technology, and deal with social issues.

RELATED WORKS:

Conventional deep learning techniques encounter numerous challenges when applied to graph data featuring a non-Euclidean structure. This has led to the introduction of graph embedding models, systematically categorized based on structural aspects, algorithmic principles, and dynamic considerations **(Lopez, 2023).** The paper provides an extensive overview of notable graph-embedding techniques, including SVD, HOPE, Node2vec, CTDNE, GCN, GraphSAGE, and GAT **(Khoshraftar & An, 2024).** The focus then shifts to Graph Neural Network (GNN) techniques, specifically GAT, GCN, and GraphSAGE, with an added enhancement through the integration of the Louvain method for community discovery **(Wang et al. 2022).** This improvement enhances model performance across diverse metrics, aligning with the evolving trends in graph neural network research.

The challenges associated with link prediction in graph data extend to metrics and classifications, especially within directed, temporal, and heterogeneous networks **(Qin & Yeung, 2023).** Various techniques, such as Hin2vec, PropFlow, GETRWR, Hin2vec, Dylink2vec, and DynamicTriad, are introduced to tackle these complexities. The significance of directional attributes, data dynamics, and dynamic features in network architectures is emphasized, highlighting endeavors to elevate prediction accuracy using a range of metrics and methodologies **(Liu et al. 2024).** In the conclusion, the study's impact on link prediction is underscored, emphasizing the establishment of a robust baseline and the utilization of comprehensive evaluation criteria as pivotal elements in the related works within this domain **(Fatouros et al. 2023).**

2.REVIEW METHODOLOGY:

The limitations and appropriate techniques for combining qualitative data in scoping reviews. It emphasises the descriptive character of scoping reviews, which aim to map out the body of existing evidence rather than reinterpret it. The article recommends against using synthesis techniques that are more suited for systematic reviews, such as thematic synthesis or meta-aggregative procedures, in scoping reviews. Rather, it supports the use of the open coding approach and basic qualitative content analysis in scoping reviews. The guidelines suggest three stages for carrying out qualitative content analysis in scoping reviews: organising, reporting, and preparation. It emphasises that abstraction is outside the purview of a scoping review because it involves synthesising or reinterpreting evidence.

LITERATURE REVIEW:

Electrical Field:

Author & Year	Main Findings	Target Application	Limitation	Proposed Theory
Sabouhi et al. (2021)	Power system vulnerability assessment.	Power systems	Ignored load importance considerations before and after disruption.	Axiomatic design-based index for topology analysis
Zhu et al. (2021)	Comprehensive weight method based on game theory	Power system reliability	Understanding and context for practical challenges and broader external factors.	Game theory-based method integrating topological structure
Yang et al. (2021)	Power system vulnerability to cascades using a graph-based model.	Enhanced resilience of power systems.	Reduce vulnerability calculation time.	Comprehensive indices based on graph theory.
Asl et al. (2020)	Integrated Energy Distribution Systems, considering unbalanced electrical networks	Integrated energy distribution systems	Comprehensive understanding of potential constraints.	Graph theory-based method for modeling energy flow
Duan, He & Zhao, (2021)	Dynamic Economic Dispatch in diverse power systems	Dynamic Economic power system	Comprehensive understanding of potential constraints.	Distributed algorithm ,Alternating Direction Method of Multipliers (ADMM)
Atkins et al. (2009)	Emphasized electrical networks' vulnerability.	Electrical Networks	Not Specified	Power grids, Transmission capacity redundancy.
Jiang et al. (2022)	Introduced a semi-decentralized	Decentralized Energy Internet	Not specified	Semi-decentralized energy

	energy routing algorithm.			
Biswas et al. (2020)	Vulnerability assessment for power systems	Power system	Fast detection of overloaded cut-sets.	Graph theory-based network analysis tool.
Han et al. (2020)	Dynamic reconfiguration of battery systems.	Battery systems.	Identified challenges in hardware design.	Reconfigurable battery systems (RBSs).
Deng, Deng & Cheong, (2021)	Improved conflict resolution in Dempster–Shafer evidence theory.	Conflict resolution in uncertain environments.	Limited discussion on the application.	Dempster–Shafer evidence theory.
Razi et al. (2020)	Proposed for residential multi-microgrid systems.	Microgrids system.	Limited discussion on the scalability.	Routing algorithm using graph theory.

Computer Science:

Author & Year	**Main Findings**	**Target Application**	**Limitation**	**Proposed Theory**
Almasan et al. (2022)	Integration of Graph Neural Networks (GNN) into Deep Reinforcement Learning (DRL)	Graph Neural Networks (GNN).	Standard DRL-based networking struggles.	GNNs into DRL Network topologies.
Casado-Vara et al. (2020)	Introduces IoT slicing, a technique combining complex networks.	Complex networks.	Limited discussion on scalability	IoT slicing technique, complex networks.
Gao (2020)	Applies link prediction to community mining using Enron e-mail data set.	Link prediction in community mining.	Limited testing on large-scale data sets, algorithm complexity.	Citation analysis, and factorization decomposition.
Anh Khoa et al. (2020)	Proposes a waste management system using IoT-based machine learning.	Waste management in university campuses.	Limited testing on large-scale data sets.	Machine learning, graph theory.
Du et al. (2020)	Utilizes NOMA and graph theory for optimization.	Nonorthogonal Multiple Access (NOMA).	Not specified	MWIS in graph theory, and a heuristic algorim.
Angel (2022)	Graph domination theory to enhance the cybersecurity in healthcare networks.	Healthcare networks.	Further details were needed for a comprehensive understanding of potential constraints.	Graph domination theory
Li et al. (2020)	Edge computing for Heterogeneous Networks (HetNets).	Heterogeneous Networks (HetNets)	Performance bounds for proposed algorithms were	Cubic Exponential Smoothing, Rapid Association (RA), Delayed Association (DA) methods,

			not derived.	JCC-UA algorithm
Krauss & McCollum (2020)	QUBO approaches for solving the shortest path problem	QUBO formulation	Quantum annealing formulations for the shortest path problem may face limitations	Directed and undirected graphs using quantum annealing.
Wang et al. (2023)	Investigated MANET transmission reliability using an SNR-Capacity model.	SNR-Capacity model	Consideration of common cause failure and cascading failure lacking.	SNR-Capacity model, Algebraic graph theory, Monte Carlo simulation
Çabuk, Tosun & Dagdeviren (2021)	Introduced MAX-Tree, a geometric tree graph for maximal area coverage in WANETs.	WANET applications such as drone networks, robotic networks.	Future work includes reducing coverage holes and exploring more connected topology formations.	MAX-Tree, a geometric tree graph for maximal area coverage in WANETs
Ogundoyin et al. (2020)	Proposed AODVRM, a secure routing protocol for MANETs.	MANETs	Limited evaluation scenarios, specific to AODV	RSA, MD5, observation-based cooperation enforcement, and graph theory-based trust techniques
Iqbal et al. (2022)	Presented an IoT-based formal model for vehicle-life interaction in VANETs using VDM-SL, UML, and graph theory.	Smart transportation systems, VANETs	Potential coverage gaps in emergency scenarios, need for further protocol design and implementation.	VDM-SL, UML, Graph Theory
Pirani, Baldi & Johansson (2022)	Detect cyber-attacks and be resilient.	Resilient Algorithms	Future research directions on robustness bounds for alternative control algorithms.	Graph Theory, Connectivity Measures, Robustness Analysis
Chen et al. (2020)	MILP to automate the assignment of mountain-valley folds.	Engineering applications, Origami design	Currently limited to structures with specific vertex degrees.	Geometric-Graph-Theoretic Representation, Mixed-Integer Linear Programming
Gao et al. (2021)	Discussed the identification of molecules in crystals using quotient graphs and reasonable decomposition of extended structures.	Materials Discovery, Crystal Structure Prediction	Future work may focus on optimizing the proposed schemes for even larger systems.	Evolution Schemes, Graph Theory, Quotient Graphs, Dimensionality Identification, Community Detection
Huang et al. (2021)	Nonlocal graph theory with a fast comprehensive alternating minimization iteration algorithm	Hyperspectral Image Classification, Remote Image Analysis	Self-tuning graph construction approaches for improved accuracy in capturing valuable similar pixels.	Transductive Learning, Variational Nonlocal Graph Theory, Sparse Graph Representation

Khaleel & Al-Shumam (2020)	Emphasizing the efficiency of Euler graph in RMI technology.	Network Security, RMI Technology	Not faster than Blowfish encryption algorithm.	Euler Graph Encoding
Godquin et al. (2020)	Deploying secure IoT services based on device capabilities.	IoT Networks, Network Security	Classifying nodes into high and low-value categories.	Weighted Graph Modeling, Dominating Sets, Centralities

Medical Applications:

Author & Year	**Main Findings**	**Target Application**	**Limitation**	**Proposed Theory**
Akbarian & Erfanian (2020)	Understanding brain structure-function relationships for reliable automatic seizure detection through EBC-based features.	Brain structure-function relationships	Limited evaluation on other databases;	MLMN, MENN, and MFNN are developed based on effective brain connectivity (EBC).
Cea-Canas et al. (2020)	Reduced theta band connectivity in both schizophrenia and bipolar disorder.	Brain networks in schizophrenia and bipolar disorder.	No significant differences observed between first-episode and chronic schizophrenia patients.	Application of graph theory measures, specifically connectivity strength (CS).
Fang et al. (2020)	Analyzed network properties in ASD children with and without regression.	ASD children with and without regression.	Sample size constraints in the ASD-R group limited additional grouping comparisons.	Utilized graph theory on DTI data.
Zhang, Chen & Lin (2021)	Optimization of Neighborhood-Based Recommendation Scheme (PPONBR) for medical-aided diagnosis and treatment.	Medical-aided diagnosis and treatment	The BGN Cryptosystem is slightly expensive, and the focus is primarily on privacy and authentication.	BGN Cryptosystem, graph theory for expanding neighbors
Bessadok, Mahjoub & Rekik (2022)	Emphasizing their role in brain graph-related tasks such as missing brain graph synthesis and disease classification.	Brain graph-related tasks	Limited exploration of predicting graphs from multiple modalities and utilizing both structural and functional connectivities.	Graph Convolution Networks (GCN) and various graph convolution operations
Jiao et al. (2020)	Hyper-graph Regularized Constrained Non-negative Matrix Factorization (HCNMF) for selecting differentially expressed genes and	Hyper-graph Regularized Constrained Non-negative Matrix Factorization (HCNMF)	Sensitivity to outliers and potential lack of robustness for bioinformatics data.	Hyper-graph regularization

	tumor classification.			
Azadifar et al. (2022)	Graph-based relevancy-redundancy gene selection method for cancer diagnosis	Cancer diagnosis	The method may have higher execution time due to considering both maximum clique and edge centrality.	Graph-based relevancy-redundancy
Barabási & Barabási (2020)	Genetic identity of neurons guides synapse and gap-junction formation.	Understanding neuronal connectivity and architecture	The model relies on the assumption that genetic factors fully explain neuronal wiring	Neurons guides synapse formation, Biclique motifs in the connetome.
Yu et al. (2020)	Hypergraph Regularized NMF (CHNMF) for clustering and feature selection on multi-cancer integrated data.	Clustering and feature selection on multi-cancer integrated data	Resulting in longer computation time compared to NMF based on the Euclidean norm.	Hypergraph regularization
Schulte-Sasse et al. (2021)	EMOGI, a graph convolutional network-based explainable machine learning method.	Prediction of cancer genes, precision oncology	Performance stability across different cancer gene datasets.	EMOGI, a graph convolutional network-based method, PPI networks
Fauchon et al. (2020)	Modular organization of pain brain networks using fMRI graph analysis informed by intracranial EEG (iEEG).	Understanding the modular organization of pain brain networks	Focused on a restricted network derived from iEEG recordings	Applied graph theory with modular analysis to fMRI data
Verma, Nagarajan & Raj (2022)	Analytic spectral graph theory-based model for capturing magnetoencephalography (MEG) frequency spectra in resting healthy subjects.	Understanding brain oscillations through spectral graph theory	The model's frequency spectra can exhibit at most two peaks	Analytic spectral graph theory-based model
Yun & Kim (2021)	Applying graph theory to diffusion tensor imaging and functional activation data in major depressive disorder (MDD).	Understanding structural and functional brain connectomes in MDD	Global network organization varied in MDD based on onset age and medication.	Graph theory to neuroimaging data.
Stumme et al. (2020)	Reorganization of RSFC in older adults particularly pertained to early sensory networks.	Functional network reorganization in older adults	High variability of RSFC alterations across older adults.	Graph theory to study RSFC in older adults

Environmental and Geographical Applications:

Author & Year	Main Findings	Target Application	Limitation	Proposed Theory
Jahanishakib et al. (2021)	Analyzed future land use scenarios for their effects on ecological	Landscape ecological networks.	Reliance on future land use scenarios introduces	Utilized graph and circuit theories

	structures.		uncertainties.	
Modica et al. (2021)	Watershed, Iran, using Graph theory to assess development impacts.	Landscape ecological networks	Limited generalizability due to the specific region	Utilized graph and circuit theories.
Mainali & Chang (2021)	Water quality within a Himalayan watershed using graph theory.	Understanding spatial patterns of dissolved oxygen (DO)	Different spatial conceptualizations in graph theory influence model strength.	Utilized graph theory and spatial regression models.
Sonnewald et al. (2020)	Systematic Aggregated Eco-Province (SAGE) method for determining global marine ecological provinces.	Understanding global marine eco-provinces.	The numerical model used may not fully capture all patterns.	SAGE method
Roy et al. (2016)	Graph theoretic network approach to model potential connectivity of natural areas.	connectivity of natural areas for invasive species.	Lacking consideration of functional aspects.	Graph theory-based network analysis
Barra et al. (2022)	Explored the use of graph metrics to characterize landscapes.	Landscape characterization	Lacks specific insights into how these metrics relate to ecological processes.	Graph metrics
Godet & Clauzel (2021)	Assess connectivity for seven pond-dwelling species in the Ile-de-France region.	Landscape connectivity for pond-dwelling species	Results were highly variable depending on the graph construction method.	Ecological networks
Chafin et al. (2021)	Explore spatio-genetic patterns and assess the impact of environmental variables on connectivity.	Genetic diversity and connectivity in dendritic ecological networks.	Limitations include potential bias from individual movements.	DISTNET and RESISTNET
Lumia et al. (2023)	Constructing an ecological network (EN) within the Reggio Calabria metropolitan area.	Ecological Network	Limitations include reliance on land use maps	Graph theory and connectivity metrics.
Pomianowski & Solon (2020)	GraphScape, a novel software system for modeling patch mosaic connectivity and ecological corridors using a patch-mosaic	Planning ecological corridors	Limitations include potential influences of large, elongated	GraphScape Software

3.REVIEW SUMMARY:

The thorough analysis highlights how graph theory may be applied to solve problems in a variety of fields, including computer science, environmental geography, power systems, and medical applications. The main goals of power systems evaluation are reliability improvement and vulnerability assessment through the use of game theory and graph theory models. In computer science, novel approaches to network optimisation and cybersecurity reinforcement are demonstrated by IoT-based machine learning, graph neural networks, and link prediction. Studies in medicine demonstrate how important graph theory is to understanding the relationships between brain shape and function, identifying cancer, and understanding how brain networks related to pain are organised.

Graph theory is important for landscape connectivity, genetic diversity, and ecological network design in environmental applications because it sheds light on ecological structures, the effects of watershed development, and spatial patterns. Overall, the synthesis highlights graph theory's wide significance and prospective research directions across disciplinary boundaries.

4.RESEARCH GAPS:

Existing research in power system vulnerability assessment, such as Sabouhi et al. (2021), may have a research gap in fully considering load importance both before and after disruptions. Future studies could focus on developing methodologies that encompass these aspects comprehensively. In computer science, the integration of Graph Neural Networks (GNN) into Deep Reinforcement Learning (DRL), as explored by Almasan et al. (2022), reveals a potential research gap in addressing scalability and performance challenges associated with incorporating GNNs into standard DRL-based networking.

While the medical field struggles to understand brain structure-function relationships, identify seizures, and be inclusive in studies of bipolar illness and schizophrenia, ecological network research faces challenges related to land use scenarios, spatial conceptualizations in graph theory that are not fully explored, and methodological variations in landscape connectivity.

Environmental studies, exemplified by Jahanishakib et al.'s (2021) analysis of future land use scenarios for ecological structures, might have a potential research gap in developing more robust models that effectively account for uncertainties introduced by varying scenarios. Addressing this gap could improve the reliability of predictions in ecological studies. Filling in these gaps is essential to knowledge advancement and making a significant contribution to each discipline.

Future research directions include improving power system vulnerability assessment, validating brain-function models across a variety of medical datasets, refining environmental models to account for uncertainties in varying scenarios for more accurate predictions, and integrating Graph Neural Networks into Deep Reinforcement Learning for improved scalability in computer science.

5.FUTURE RESEARCH DIRECTION:

The proposals emphasise important areas for future research in a variety of fields. In power systems, improved optimisation integration with economic dispatch and load importance for dynamic vulnerability evaluations are prioritised. It is critical to close gaps in cut-set detection, microgrid routing scalability, and decentralised energy internet optimisation. Priorities in computer science include creating reliable VANET protocols, improving WANET connectivity, and resolving scalability issues in IoT slicing. In the medical domain, thorough database searches for brain connection, inclusion in research of bipolar disorder and schizophrenia, and reasonably priced substitutes for medically assisted diagnosis are critical. The need of addressing uncertainties in land use scenarios, investigating spatial conceptualizations, and taking methodological variances into account for landscape connectivity—especially in the vicinity of ponds—is highlighted by ecological network research.

6.CONCLUSION:

Graph theory is a highly adaptable and potent instrument that finds extensive use in domains like computer science, environmental studies, power systems, and medicine. It helps with understanding brain functions and supports cancer diagnosis in medical research; it addresses vulnerabilities and improves reliability in power systems; it contributes to innovations like graph neural networks in computer science; and it offers insights into ecological structures and spatial patterns in environmental studies. This synthesis emphasises how important graph theory is to many other fields and how it provides a fundamental basis for multidisciplinary understanding of complex systems.

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