The class $D(T^k)$ −operators

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Abstract: *This article presents a new category of operators called $D(T^k)-$ operators, which operate on a complex Hilbert space H. An operator $T \in B(H)$ is considered a $D(T^k)-$ operators if the equation $(T^* T)^k U = U(T^*T)^k$ holds, where k is a positive integer greater than 1, and T^* is the adjoint of the operator T, We will explore the fundamental characteristics of these operators and provide examples for better understanding.*

Keywords: Hilbert space, normal operators, $D(T)$ −operators, adjoint operators, bounded linear operators.

1-Introduction

In this article, $B(H)$ refers to the algebra consisting of all bounded linear operators on a complex Hilbert space H. A Hilbert space is a mathematical space that has an inner product and is also characterized by its completeness with respect to the norm induced by this inner product. An operator T^* is defined as the adjoint of T if and only if the inner product (Tx, y) is equivalent to (x, T^*y) for all x and y belonging to the set H. A normal operator , which maps from a Hilbert space H to itself, is defined by the equation $T^*T = TT^*$. The theory of operators in Hilbert space has been thoroughly analysed by several authors, as seen by the references [1, 2, 3]. In 2021, Elaf. S. A. conducted an examination of the class of operators $D(T)$ as described in [4], and presented the fundamental characteristics of this class in a Hilbert space. An operator $T \in B(H)$ is referred to as a $D(T)$ −operators if there exists $U \in B(H)$ such that $U \neq 0, I$ and $T^*TT = TT^*T$.

This article presents the class $D(T^k)-$ operators as an extension of the class $D(T)$ operators and examines its essential characteristics. We shall establish the conditions under which an operator T is considered to be $D(T^k)$. To be classified in this category, the addition and multiplication of two operators $D(T^k)$ must meet certain conditions, which we will examine.

The representation of any operator T in cartesian form is $T = A + Bi$, where A and B represent the real and imaginary components of T, respectively. The real component of T, represented as $Re\, T$, is calculated as the average of T and its complex conjugate, which is $\frac{T+T^*}{2}$.On the other hand, the imaginary component of T, written as $Im\, T$, is determined by taking the differences between T and its complex conjugate, divided by $2i$, resulting in $\frac{T-T^*}{2i}$.

2-Main Results

The objective of the work is to introduce a new category of operators known as $D(T^k)-$ operators and analyze fundamental properties of this category.

2-1 Definition:
Let T be a bounded operator from a complex Hilbert space H to itself, the T is said to be a class $D(T^k)-$ operators if there exists is an operator $U \in B(H)$ such that $U\left({T^*}^k T^k\right) = \left({T^*}^k T^k\right)U$, where k is a positive integer greater than 1.

2-2 Example:

The operators T and U are two operators in the two-dimensional Hilbert space $\mathbb{C}^2$,
where $T = \begin{bmatrix} 2i & 2 \\ 0 & -2i \end{bmatrix}$ and $U = \begin{bmatrix} 0 & -i \\ i & 1 \end{bmatrix}$

$$U({T^*}^2 T^2) = \begin{bmatrix} 0 & -4i \\ 4i & 4 \end{bmatrix} = \begin{bmatrix} 0 & -4i \\ 4i & 4 \end{bmatrix} = ({T^*}^2 T^2)U.$$

So $T \in D(T^2)-$ operators.

If k is equal to 3, then $U\left(T^{*3}\,T^{3}\right) = \begin{bmatrix} -64 & 0 \\ 0 & 64 \end{bmatrix} \neq \begin{bmatrix} 64 & -128i \\ 0 & 64 \end{bmatrix} = \left(T^{*3}\,T^{3}\right)U.$
Therefore, T is not a $D(T^3)$ operator.

If the operator $T \in D(T^k) -$ operators, it is not necessarily an operator in $D(T^{k+1})$ for $k > 1$.

The following statements may be deduced from the definitions of the normal operator and the $D(T^k) -$ operators.

2-3 <u>Remarks</u>:
1. When the value of k=1, it follows that T is a $D(T) -$ operator.
2. $If\ T\ is\ a\ normal\ operator, then\ both\ T\ and\ T^*\ are\ D(T^k\ ) - operators.$

The subsequent theorem shows a link between the $D(T) -$ *and* $D(T^k) -$ operators of the two classes.

2-3 <u>Theorem</u>:
If $T \in D(T)$ −operators, then $T \in D(T^k) -$ operators for each $k > 1$.

Proof:

$$U\left(T^{*^k}T^k\right) = U\left(T^{*^{k-1}}(T^*T)T^{k-1}\right)$$
$$= U(T^{*^{k-1}}(T^{k-1}(T^*T)),$$

since $T \in D(T)$

$$= U(T^{*^{k-2}}(T^*T)T^{k-2}(T^*T)$$
$$\vdots$$
$$= U(T^*T)(T^*T)\cdots\cdots(T^*T).$$

$$\left(T^{*^k}T^k\right)U = \left(T^{*^{k-1}}(T^*T)T^{k-1}\right)U$$
$$= (T^{*^{k-1}}(T^{k-1}(T^*T))U,$$

since $T \in D(T)$

$$= (T^{*^{k-2}}(T^*T)T^{k-2}(T^*T)U$$
$$\vdots$$
$$= (T^*T)(T^*T)\cdots\cdots(T^*T)U.$$
$$= U(T^*T)(T^*T)\cdots\cdots(T^*T).$$

Therefore, we conclude that T is an operator belonging to the $D(T^k)$ class.

The example below demonstrates that the converse of the above claim is not true.

2-4 <u>Example</u>:
Consider the operators $T = \begin{bmatrix} 2i & 2 \\ 0 & -2i \end{bmatrix}$ and $U = \begin{bmatrix} 0 & -i \\ i & 1 \end{bmatrix}$ in the two-dimensional Hilbert space $₵^2$.
The operator T belongs to the class $D(T^k)$ where $U(T^*T)^2 = (T^*T)^2U$ but $U(T^*T) \neq (T^*T)U$.
Thus, T is not a member of class $D(T)$.

The operator T can be classified as a member of class D (T^k) by satisfying particular conditions, as proven by the following theorem.

2-5 <u>Theorem</u>:
An operator T belongs to the class $D(T^k)$ if and only if T^*T commutes with $Re\ U$ and $Im\ U$.

Proof:
Suppose T is an element of $D(T^k)$, i.e., $U(T^*T)^k = (T^*T)^kU$, then $T^*T\ Re\ U = Re\ U\ T^*T$ and $T^*T\ Im\ U = Im\ U\ T^*T$.
On the other hand, the equation $T^*T\ Re\ U = Re\ U\ T^*T$ and $T^*T\ Im\ U = Im\ U\ T^*T$ hold true.
Hence, $T^*T[Re\ U + i\ Im\ U] = [Re\ U + i\ Im\ U]T^*T and\ we\ have\ T^*TU = UT^*T$.

$So, U(T^*T)^k = (T^*T)^k U.$

2-6 Propositions:
Let $T: H \to H$ be a $D(T^k)$ − operator, then
1. T^m also belongs to the class of $D(T^k)$ − operators, where $m > 1$.
2. λT belongs to the class $D(T^k)$ − operators with $\lambda \in \mathcal{R}$.
3. If T^{-1} exists, then T^{-1} belongs to the class $D(T^k)$ − operators.

Proof:
1- To demonstrate that T^m is an operator inside the class of $D(T^k)$ − operators. We will now proceed with the induction on the variable m. If $m = 1$, then the result is true, so more proof is unnecessary.
Let us assume that the outcome ''true'' is valid for $m = p$.

$$\left(U\left(T^{*^k}T^k\right)\right)^p = \left(\left(T^{*^k}T^k\right)U\right)^p$$

Following that, we will prove that it holds true when $m = p + 1$.

$$\begin{aligned}\left(U\left(T^{*^k}T^k\right)\right)^{p+1} &= \left(U\left(T^{*^k}T^k\right)\right)^p U\left(T^{*^k}T^k\right)\\ &= \left(\left(T^{*^k}T^k\right)U\right)^p \left(T^{*^k}T^k\right)U\\ &= \left(\left(T^{*^k}T^k\right)\right)^{p+1} U^{p+1}\\ &= \left(\left(T^{*^k}T^k\right)U\right)^{p+1}.\end{aligned}$$

The induction proof is now completed.
Therefore, $T^m \in D(T^k)$ − operators.

The subsequent assertions in this statement may be demonstrated straightforwardly based on the definition of the $D(T^k)$ − operators as follows:

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$$\begin{aligned}(\mu T^*)^k(\mu T)^k U &= \mu^k (T^*)^k \mu^k (T)^k U\\ &= \mu^k \mu^k (T^*)^k (T)^k U\\ &= \mu^k \mu^k U (T^*)^k (T)^k\\ &= U(\mu T^*)^k (\mu T)^k.\end{aligned}$$

3-

$$\begin{aligned}\left(T^{-1^*}\right)^k (T^{-1})^k U &= \left(T^{*^{-1}}\right)^k (T^{-1})^k U\\ &= \left(T^{*^{-1}}T^{-1}\right)^k U\\ &= U\left(T^{*^{-1}}T^{-1}\right)^k\\ &= U\left(T^{*^{-1}}\right)^k (T^{-1})^k\\ &= U\left(T^{-1^*}\right)^k (T^{-1})^k.\end{aligned}$$

2-7 Theorem:
If T and U are invertible operators and $T \in D(T^k)$ operators, then T^{-1^*} also belongs to the class $D(T^k)$ − operators.
Proof:

$$U\left(T^{*^k}T^k\right) = \left(T^{*^k}T^k\right)U$$
$$T^{-1^k}T^{-1^{*^k}}U^{-1} = U^{-1}T^{-1^k}T^{-1^{*^k}}$$
$$U^{-1^*}\left(T^{-1^k}T^{-1^{*^k}}\right) = \left(T^{-1^k}T^{-1^{*^k}}\right)U^{-1^*}$$

Therefore, T^{-1^*} belongs to the class of $D(T^k)-$ operators.

3-Properties of $D(T^k)$ −operators.

This section focuses on analyzing algebraic characteristics that are associated with our class of $D(T^k)-$ operators.

3-1 Remark:
If T_1, T_2 and U are operators belonging to the class of $D(T^k)-$ operators, it is not certain that T_1+T_2 will also belong to be $D(T^k)$. The following examples can help to illustrate this.

3-2 Example:

Let $T_1=\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}, T_2=\begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}$ are two operators of class $D(T^2)$and $U=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\in B(H)$.

$$T_1+T_2=\begin{bmatrix}0 & 3\\ 3 & 1\end{bmatrix}$$

$$\left((T_1+T_2)^*(T_1+T_2)\right)^2 U=\begin{bmatrix}81 & 81\\ 99 & 99\end{bmatrix}\neq\begin{bmatrix}81 & 99\\ 0 & 99\end{bmatrix}=U\left((T_1+T_2)^*(T_1+T_2)\right)^2.$$

3-3 Example:

The operators $T_1=\begin{bmatrix}1 & 0 & i\\ 0 & 1 & 0\\ -i & 0 & 1\end{bmatrix}$ and $T_2=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -i\\ i & 0 & 1\end{bmatrix}$ are two Hilbert space $\mathbb{C}^3$ which are in the class $D(T^k)$ −operators with a bounded linear operator $U=\begin{bmatrix}1 & 0 & -i\\ 0 & 0 & 0\\ 1 & 0 & i\end{bmatrix}$ but not T_1+T_2.

Now, the following theorem holds true if the conditions necessary for remark (3-1) are satisfied.

3-6 Theorem:
Let T_1 and T_2 be two $D(T^k)-$ operators on a Hilbert space H such that $T_1T_2=T_1^*T_2^*=T_2^*T_1^*=0$ then T_1+T_2 is a $D(T^k)$ −operator.

Proof:

$$(T_1+T_2)^{*k}(T_1+T_2)^k U=(T_1^*+T_2^*)^k(T_1+T_2)^k U$$
$$=\left(T_1^{*k}+kT_1^{*k-1}T_2^*+\cdots+T_2^{*k}\right)\left(T_1^k+kT_1^{k-1}T_2+\cdots+T_2^{k}\right)U$$
$$=\left(T_1^{*k}+T_2^{*k}\right)\left(T_1^k+T_2^k\right)U$$
$$=\left(T_1^{*k}T_1^k+T_1^{*k}T_2^k+T_2^{*k}T_1^k+T_2^{*k}T_2^k\right)U$$

Since T_1 and T_2 are two $D(T^k)-$ operators.

$$=U\left(T_1^{*k}T_1^k+T_2^{*k}T_2^k\right)$$
$$\vdots$$
$$=U(T_1+T_2)^{*k}(T_1+T_2)^k.$$

Hence, T_1+T_2 is a $D(T^k)$ −operator.

Moreover, it is important to mention that the product of T_1 and T_2 may not always be a D (T^k)-operator.

Therefore, it can be proven that by examining the example of (3-2):

$$(T_1T_2)^*(T_1T_2)U=\begin{bmatrix}-7 & -17\\ 5 & 3\end{bmatrix}\neq\begin{bmatrix}-2 & -12\\ 5 & -2\end{bmatrix}=U(T_1T_2)^*(T_1T_2)$$

The subsequent theorem establishes that, under certain conditions, the multiplication of two operators inside the specified class is likewise a member of this class.

3-7 Theorem:
Consider T_1 and T_2 as two $D(T^k) -$ operators on a Hilbert space H, where $T_1T_2 = T_2T_1$ and $T_2T_1^* = T_1^*T_2$. In this case, it can be concluded that T_1T_2 is also a $D(T^k)$ operator.

Proof:

$$\begin{aligned}(T_1T_2)^{*k}(T_1T_2)^kU &= (T_2^*T_1^*)^k(T_1T_2)^kU\\ &= T_2^{*k}T_1^{*k}T_2^kT_1^kU\\ &= T_2^{*k}T_2^kT_1^{*k}T_1^kU\\ &=T_2^{*k}T_2^kUT_1^{*k}T_1^k\\ &=UT_2^{*k}T_2^kT_1^{*k}T_1^k\\ &=U(T_1T_2)^{*k}(T_1T_2)^k.\end{aligned}$$

Thus, T_1T_2 is a $D(T^k)$ −operator.

4- Conclusion:

This paper introduces a new class of operators known as $D(T^k) -$ operators and examines its features through illustrative examples. The paper presents the primary findings, which are:

1. If $T \in D(T)$ −operators, then $T \in D(T^k) -$ operators for every $k > 1$.
2. Consider T_1 and T_2 be two $D(T^k) -$ operators on a Hilbert space H. If $T_1T_2 = T_1^*T_2^* = T_2^*T_1^* = 0, then$ $T_1 + T_2$ is also a $D(T^k)$ −operator.
3. Consider T_1 and T_2 be two $D(T^k) -$ operators on a Hilbert space H. If $T_1T_2 = T_2T_1$ and $T_2T_1^* = T_1^*T_2$,then T_1T_2 is also a $D(T^k)$ −operator.

4-References:

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