Solving Linear Differential Equations with Constant Coefficients

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Abstract - *In this paper, we are discussing about the different method of finding the solution of the linear differential equations with constant coefficient. For more understanding of different method we have included different examples to find Particular solution of function of linear differential equations with constant coefficient. Particular solution is combination of complementary function and particular integral.*

Key Words: Particular Integral, Complementary Function, Auxiliary equation.

1. INTRODUCTION

This paper includes overview of the definition and different types of finding solution of the linear differential equations with constant coefficient. Here we mentioned different rules for finding particular solution. We applied the linear differential equation in various science and engineering field. It is applied to electrical circuit, chemical kinetics, Heat transfer, control system, mechanical systems etc. First we see the definitions of linear differential equation with constant coefficient then we see general solution with complementary function and particular integral.

1.1 Linear Differential Equation

Definition:

A general linear differential equation of nth order with constant coefficients is given by:

$$k_0\frac{d^ny}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+---+k_{n-1}\frac{dy}{dx}+k_ny=F(x)$$

Where K's are constant and F(x) is function of x alone or constant.

$$(k_0D^n+k_1D^{n-1}+--+k_{n-1}D+k_n)y=F(x)$$ Or $$f(D)y=F(x)$$

1.2 Solving Linear Differential Equations with Constant Coefficients:

Complete solution of equation f(D)y=F(x) is given by C.F + P.I. Where C.F. denotes complementary function and P.I. is particular integral. When F(x) =0, then the solution of equation F (D)y=0 is given by y=C.F.

Otherwise the solution of the F(D)y=X is satisfies the particular integral.

The complete solution of the differential equation is

Y= C.F. + P.I.

i.e. Y=Complementary Function + Particular integral

2. Methods of Finding Complementary Function

Consider the equation F(D)y=F(x)

$$(k_0D^n+k_1D^{n-1}+--+k_{n-1}D+k_n)y=F(x)$$

Step 1: Put D=m, Auxiliary equation is given by f(m)=0

$$(k_0m^n+k_1m^{n-1}+---+k_{n-1}m+k_n)y=0$$

Step 2: Solve the auxiliary equation and find the roots of the equation. Lets roots are $m_1,m_2,m_3\ldots m_n$.

Step 3: After we get complementary function according to given rules below:

Table -1: Methods of finding C.F.

Methods of finding C.F	
Roots of A.E.	Complementary function (C.F.)
If the n roots of A.E. are real and distinct say $m_1=m_2=\cdots=m_n$	$C_1e^{m_1x}+C_2e^{m_2x}+\ldots\ldots+C_ne^{m_nx}$
If two or more roots are equal.i.e. $m_1=m_2=\cdots=m_k$.	$(C_1+C_2x+C_3x^2+\cdots+C_kx^{k-1})e^{m_1x}+\ldots\ldots+C_ne^{m_nx}$
If roots are imaginary $m_1=\alpha+i\beta$, $m_2=\alpha-i\beta$	$e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$
If roots are imaginary and repeated twice $m_1=m_2=\alpha+i\beta$, $m_3=m_4=\alpha-i\beta$,	$e^{\alpha x}[(c_1+c_2x)\cos\beta x+(c_3+c_{4x})\sin\beta x)]+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$

Exaple 1] Solve the differential equation:

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Ans : The given differential equation is

$$(D^3 + 2D^2 + D)y = 0$$

Now put D=m then auxiliary equation is

$$m^3 + 2m^2 + m = 0$$

$$m(m^2 + 2m + 1) = 0$$

$$m(m+1)^2 = 0$$

$$m = 0, -1, -1$$

Here roots are 0,-1,-1 then C.F. is

$$\text{C.F.} = c_1x + (c_2 + c_3x)e^{-x}$$

Since F(x) =0 then solution is given by Y=C.F.

$$Y = c_1x + (c_2 + c_3x)e^{-x}$$

Exaple 2] Solve the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

Ans :

The given differential equation is

Now put D=m then auxiliary equation is

$$m^2 - 8m + 15 = 0$$

$$(m-3)(m-5) = 0$$

$$m = 3{,}5$$

Here roots are 3 & 5 then we have C.F. is

$$\text{C.F.} = c_1e^{3x} + c_2e^{5x}$$

Since F(x) =0 then solution is given by Y=C.F.

$$Y = c_1e^{3x} + c_2e^{5x}$$

3 Short Methods for Finding Particular integral

If X ≠ 0, in equation (1) then

$$\text{P.I.} = \frac{1}{f(D)}X$$

Clearly P.I. =0 if F(x) =0

Following are the methods for finding particular integral:

Table-2: **Rules for finding P.I.**

Types of function	What to do	Corresponding P.I.
X = e^{ax}	Put D = a in f(D)	$\text{P.I.} = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}\quad f(a) \neq 0$ In case of failure, i.e. f(a)=0 $\text{P.I.} = x\frac{1}{f'(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}\quad f'(a) \neq 0$ If f'(a)=0 $\text{P.I.} = x^2\frac{1}{f''(a)}e^{ax}\quad f''(a) \neq 0$ And so on
X = (sin ax+b) or (cos ax+b)	Put D^2= $-a^2$ $D^3=D.D^2$ $=-a.D$ $D^4=(D^2)^2$ $=a^4$ ... so on in f(D)	This will form a linear expression in the denominator. Now rationalize the denominator to substitute. Operate on the numerator term by term by taking $D = \frac{d}{dx}$ If case fail ,then do as type 1
X = x^n	Take $[F(D)]^{-1}$ x^m	Expand $[F(D)]^{-1}$ using binomial expansions and if (D-a) remains in the denominator then take rationalization of denominator. Treat D in the numerator as derivative of the corresponding function
X = $e^{ax}g(x)$ Where g(x) is any function of x	Make f(D) is f(D+a)	We have to need $\frac{1}{f(D)}e^{ax}g(x) = e^{ax}\left(\frac{1}{f(D+a)}g(x)\right)$

3.1 Solved Example

Example 1] Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$$

Solution:

The given differential equation is

$$(D^2 - 2D + 10)y = 0$$

The auxiliary equation is

$$m^2 - 2m + 10 = 0$$

$$m = 1 \pm 3i$$

Here roots are 1+3i and 1-3i

Then

$$\text{C.F.} = e^x(c_1\cos 3x + c_2\sin 3x)$$

Now we see P.I.

$$P.I. = \frac{1}{f(D)} F(x)$$

$$P.I. = \frac{1}{f(D)} e^{2x}$$

$$= \frac{1}{f(2)} e^{2x}$$

By putting $D = 2$

$$= \frac{1}{2^2 - 2(2) + 10} e^{2x}$$

$$= \frac{1}{10} e^{2x}$$

Then complete solution is Y=C.F.+P.I.

$$Y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10} e^{2x}$$

Ex.2] Solve the differential equation $\frac{d^2y}{dy} - y = 5x - 2$

Solution:

The differential equation is

$(D^2-1)y=5x-2$

The auxiliary equation is

$m^2-1=0$

$m=\pm 1$

The roots are +1,-1

Then the C.F. is

C.F.=$c_1 e^x + c_2 e^{-x}$

Then

$$P.I. = \frac{1}{f(D)} F(x)$$

$$= \frac{1}{D^2-1}(5x-2)$$

$$= \frac{1}{-1(1-D^2)}(5x-2)$$

$$= -(1-D^2)^{-1}(5x-2)$$

$$= -(1+D^2+\cdots)\ (5x-2)$$

$$= -(5x-2)$$

$$P.I. = -5x + 2$$

Then complete solution is Y=C.F. +P.I.

Thus Y= $c_1 e^x + c_2 e^{-x} - 5x + 2$

3. CONCLUSIONS

In this paper we conclude that the solution of the differential equation is linear combination of the homogenous and particular solutions. The brief study of the linear differential equation with constant coefficient provides students understandable data in summaries form to apply this data in the field of physics, biology, and engineering. They also used into the electric circuits, mechanical vibrations, population growths and chemical reactions.

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