ON SOME FIXED POINT RESULTS IN GENERALIZED METRIC SPACE WITH SELF MAPPINGS UNDER THE BOUNDS

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Abstract

Generalized metric spaces are important in many fields and are regarded as mathematical tools. The idea of generalized metric space is introduced in this study, and various sequence convergence qualities are demonstrated. We also go over the continuous and self mappings fixed point extended result.

Keyword: Generalized metric spaces, Continuous mappings, Self mappings, Fixed point theory.

1. INTRODUCTION

The study of fixed point theory has been at the center of vigorous activity, although they arise in many other areas of mathematics. In 1992, Dhage [1] developed the concept of generalized metric space, often known as D-metric space, and demonstrated the existence of a single fixed point for a self-map that satisfies a contractive condition. Rhoades [4] discovered certain fixed point theorems and generalized Dhage's contractive condition. The Rhoades contractive condition was also extended by Dhage's [3] to two maps in D-metric space. Dhage [2] discovered a singular common fixed point [6] on a D-metric space by applying the idea of weak compatibility of self-mappings.

A generalized metric on set X is a function $D: X \times X \times X \to R_+$ such that for any $x, y, z, w \in X$ (i) $D(x, y, z) \geq 0$ and $D(x, y, z) = 0$ if and only if $x = y = z$, (ii) $D(x, y, z) = D\big(p(y, x, z)\big)$, where p is a permutation, and (iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$. The pair (X, D) is referred to be generalized metric space after that. A triangle with the vertices x, y and z has a peridiameter defined by a generalized metric $D(x, y, z)$.

Definition 1.1 If $R(a, b, c) = \max(a, b, c)$ then $D(x, y, z)(p + q + r) \leq R(D(x, y, a)(p), D(x, a$

$, z)(q), D(a, y, z)(r)\big) \Leftrightarrow \tau_a(x, y, z) \leq \tau_a(x, y, a) + \tau_a(x, a, z) + \tau_a(a, y, z)$ for all $\alpha \in [0,1]$ and $x, y, z \in X$.

Definition 1.2 If $S(a, b, c) = \min(a, b, c)$ then $D(x, y, z)(p + q + r) \geq S(D(x, y, a)(p), D(x, a$

$, z)(q), D(a, y, z)(r)\big) \Leftrightarrow \varepsilon_a(x, y, z) \leq \varepsilon_a(x, y, a) + \varepsilon_a(x, a, z) + \varepsilon_a(a, y, z)$ for all $\alpha \in [0,1]$ and $x, y, z, a \in X$.

Definition 1.3 A sequence $\{x_n\}$ in a D-metric space is said to be Cauchy if for any given $\epsilon > 0$, there exists n_0 such that for all $r, s, t > n_0$, $D(x_r, x_s, x_t) < \epsilon$.

Definition 1.4 f is said to be orbitally continuous if for each $x \in X$, $\{x_n\} \subset O_f(x)$, $x_n \to x^* \Rightarrow$

$f x_n \to f x^*. O_f(x) = \{x\} \cup \{f^n x : n \in N\}$.

Theorem 1.5 Let (X, D) be a complete bounded D-metric space and T be a self map of X satisfying the condition that if there exists a $k \in [0, 1)$ such that for all $x, y, z \in X$ if

$D(Tx, Ty, Tz) \leq k \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}$.

Then T has a unique fixed point u in X and T is continuous at u.

Theorem 1.6 Let (X, D) be a compact D-metric space and T be a continuous self map of X satisfying for all $x, y, z \in X$ with $D(x, y, z) \neq 0$,

$D(Tx, Ty, Tz) < \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}$.

Then T has a unique fixed point $u \in X$.

Definition 1.7 Let T represent a multi-valued map [7] on the (X, D) D-metric space. Let $x_0 \in X$. If $x_{n-1} \in T^{n-1}(x_0)$, then $x_n \in Tx_{n-1}, \forall n \in N$, then a sequence $\{x_n\}$ in X is said to represent an orbit of T at x_0 denoted by $O(T, x_0)$. If an orbit's diameter is finite, it is said to be bounded. If every Cauchy sequence in it converges to a point on X, then it is said to be complete.

$D(x, y, z) = \max[\emptyset(x, y), \emptyset(x, z), \emptyset(y, z)]$ is an example [5] of D-metric., where $\emptyset$ is a metric on X, and (b) $D(x, y, z) = \emptyset(x, y) + \emptyset(x, z) + \emptyset(y, z)$.

When a sequence $\{x_n\}$, $\lim_{n\to\infty} x_n = x$ in a D-metric space (X, D) converges to a point x, it is said to be D-convergent. If there is an $n_0 \in N$ such that $D(x_n, x_m, x) < \delta\ \forall n, m > n_0$.

If $\emptyset_m[(s_1, s_2, s_3, s_4, s_5), s_6] = \emptyset[\max(s_1, s_2, s_3, s_4, s_5)] - s_6$ and $\emptyset_m: R_+^5 \times R_+ \to R_+$ In this case, $\emptyset: R_+ \to R_+$ is a non-decreasing upper semi-continuous function with $\emptyset(0) = 0$ and $\emptyset(s) < s$ for $s > 0$. Then, on $R_+^5 \times R_+$, $\emptyset_m$ is upper semi-continuous, and $\emptyset_m$ is non-decreasing on R_+^5. Additionally, $\emptyset_m[(p, p, p, p, p), q] \geq 0 \Rightarrow q \leq \emptyset(p)$. Therefore, $\emptyset_m \in \phi$.

2. OUR RESULTS

Theorem 2.1 If (X, D) is a complete generalized metric space with three self-mappings of X that satisfy the criterion T_1, T_2 and T_3 such that

$$D(T_1x, T_2y, T_3z) \leq \beta_1 \max[D(y, T_1x, T_2y), D(y, T_2y, T_3z), D(y, T_3z, T_1x)]$$
$$+\beta_2 \max[D(x, T_2y, T_3z), D(x, y, z), D(y, T_2y, T_3z)] + \beta_3 D(x, T_1x, T_2y)$$

For all $x, y, z \in X$ and $\beta_1, \beta_2, \beta_3 \in R$ satisfying the condition $3\beta_1 + 3\beta_2 < 1 - \beta_3$. Then T has fixed point.

Proof: Let $x_0 \in X$ and define a sequence $\{x_n\}$ of points of $X: T_1x_{2n} = x_{2n+1}, T_2x_{2n+1} = x_{2n+2}, T_3x_{2n+2} = x_{2n+3}, n = 0,1, \ldots \ldots$ Now, applying the given condition we achieve the result as follows $D(T_1x_{2n}, T_2x_{2n+1}, T_3x_{2n+2}) = D(x_{2n+1}, x_{2n+2}, x_{2n+3})$

$\leq \beta_1 \max[D(x_{2n+1}, x_{2n+1}, x_{2n+2}), D(x_{2n+1}, x_{2n+2}, x_{2n+3}), D(x_{2n+1}, x_{2n+3}, x_{2n+1})] +$

$\beta_2 \max[D(x_{2n}, x_{2n+2}, x_{2n+3}), D(x_{2n}, x_{2n+1}, x_{2n+2}), D(x_{2n+1}, x_{2n+2}, x_{2n+3})] +$

$\beta_3 D(x_{2n}, x_{2n+1}, x_{2n+2})$

$i.e.\ D(x_{2n+1}, x_{2n+2}, x_{2n+3}) \leq \beta_1 D(x_{2n+1}, x_{2n+1}, x_{2n+2}) + \beta_2 D(x_{2n}, x_{2n+2}, x_{2n+3}) +$

$\beta_3 D(x_{2n}, x_{2n+1}, x_{2n+2})$

$i.e.\ D(x_{2n+1}, x_{2n+2}, x_{2n+3})$

$$\leq \beta_1[D(x_{2n+3}, x_{2n+1}, x_{2n+2}) + D(x_{2n+1}, x_{2n+3}, x_{2n+2}) + D(x_{2n+1}, x_{2n+1}, x_{2n+3})]$$
$$+\beta_2[D(x_{2n+1}, x_{2n+2}, x_{2n+3}) + D(x_{2n}, x_{2n+1}, x_{2n+3}) + D(x_{2n}, x_{2n+2}, x_{2n+1})]$$
$$+\beta_3 D(x_{2n}, x_{2n+1}, x_{2n+2})$$

$i.e.\ D(x_{2n+1}, x_{2n+2}, x_{2n+3})[1 - 2\beta_1 - \beta_2]$

$$\leq [\beta_1 + 2\beta_2 + \beta_3] \max \begin{bmatrix} D(x_{2n+1}, x_{2n+1}, x_{2n+3}), D(x_{2n}, x_{2n+1}, x_{2n+3}) \\ D(x_{2n}, x_{2n+2}, x_{2n+1}), D(x_{2n}, x_{2n+1}, x_{2n+2}) \end{bmatrix}$$

$i.e.\, D(x_{2n+1}, x_{2n+2}, x_{2n+3})$

$$\leq p \max \begin{bmatrix} D(x_{2n+1}, x_{2n+1}, x_{2n+3}), D(x_{2n}, x_{2n+1}, x_{2n+3}) \\ D(x_{2n}, x_{2n+2}, x_{2n+1}), D(x_{2n}, x_{2n+1}, x_{2n+2}) \end{bmatrix}, \text{here } p = {\beta_1 + 2\beta_2 + \beta_3}/{1 - 2\beta_1 - \beta_2}$$

But we have to given that $3\beta_1 + 3\beta_2 < 1 - \beta_3$ so $p = {\beta_1 + 2\beta_2 + \beta_3}/{1 - 2\beta_1 - \beta_2} < 1$

Letting $q = \max \begin{bmatrix} D(x_{2n+1}, x_{2n+1}, x_{2n+3}), D(x_{2n}, x_{2n+1}, x_{2n+3}) \\ D(x_{2n}, x_{2n+2}, x_{2n+1}), D(x_{2n}, x_{2n+1}, x_{2n+2}) \end{bmatrix}$. If $q = D(x_{2n+1}, x_{2n+2}, x_{2n+3})$ then we achieve $D(x_{2n+1}, x_{2n+2}, x_{2n+3}) \leq pD(x_{2n+1}, x_{2n+1}, x_{2n+3})$ and this implies $D(x_{2n+1}, x_{2n+2}, x_{2n+3}) \leq pD(x_{2n}, x_{2n}, x_{2n+3})$ similarly we can obtain $D(x_{2n+1}, x_{2n+2}, x_{2n+3})$

$< pD(x_{2n}, x_{2n}, x_{2n+2}) < p^2 D(x_{2n-1}, x_{2n-1}, x_{2n+2}) < \cdots < p^2 D(x_1, x_1, x_3)$. Now when we have $q = D(x_{2n}, x_{2n+1}, x_{2n+3})$ then we get $D(x_{2n+1}, x_{2n+2}, x_{2n+3}) \leq pD(x_{2n}, x_{2n+1}, x_{2n+3})$ and hence $D(x_{2n}, x_{2n+1}, x_{2n+2}) \leq pD(x_{2n-1}, x_{2n}, x_{2n+2})$. So by symmetry $D(x_{2n+1}, x_{2n+2}, x_{2n+3})$

$< pD(x_{2n}, x_{2n+1}, x_{2n+3}) < p^2 D(x_{2n-1}, x_{2n}, x_{2n+1}) < \cdots < p^{2n+1} D(x_0, x_1, x_3)$. Again, we have $q = D(x_{2n}, x_{2n+2}, x_{2n+1})$ then we get $D(x_{2n+1}, x_{2n+2}, x_{2n+3}) < p^{2n+1} D(x_0, x_2, x_1)$ and which shows that $\{x_n\}$ is a Cauchy sequence because $p < 1$. Hence there exists some point $u \in X$ such that $\lim_{n\to\infty} x_n = u$ because X is complete. Now we shall show that u is the common fixed point of T_1, T_2 and T_3 $D(T_1 u, T_2 u, T_3 x_{2n+3}) = D(T_1 u, T_2 u, T_3 x_{2n+2})$

$$\leq \beta_1 \max[D(u, T_1 u, T_2 u), D(u, T_2 u, T_3 x_{2n+3}), D(u, T_3 x_{2n+3}, T_1 u)] +$$
$$\beta_2 \max[D(u, T_2 u, T_3 x_{2n+3}), D(u, u, x_{2n+2}), D(u, T_2 u, T_3 x_{2n+3})] +$$
$$\beta_3 D(u, T_1 u, T_2 u)$$
$$\leq \beta_1 \max[D(u, T_1 u, T_2 u), D(u, T_2 u, x_{2n+3}), D(u, x_{2n+3}, T_1 u)] +$$
$$\beta_2 \max[D(u, T_2 u, x_{2n+3}), D(u, u, x_{2n+2}), D(u, T_2 u, T_3 x_{2n+3})] +$$
$$\beta_3 D(u, T_1 u, T_2 u)$$

Now, when $n \to \infty$ we achieve $D(T_1 u, T_2 u, u) \leq 0$ and this shows that $T_1 u = T_2 u = u$ by symmetry we also achieve $T_2 u = T_3 u = u$. Thus, $T_1 u = T_2 u = T_3 u$.

Theorem 2.2 If T meets the criteria, let T be an orbitally continuous mapping of a bounded complete D-metric space X into itself such that

(i) $x_n \to x, y_n \to y, z_n \to z$ in X then $D(x_n, y_n, z_n) \to D(x, y, z)$

$$(ii)\ D(Tx, Ty, Tz) \leq \alpha_1 D(x, y, Tz) + \alpha_2 \max\{D(x, Ty, Tz), D(y, Tx, Tz)\} +$$
$$\alpha_3 \max\{D(x, Ty, Tz), D(y, Tx, Tz)\} + \alpha_4 \max\{D(Ty, T^2 x, Tz), D(x, T^2 x, Tz)\}$$
$$+\alpha_5 \max\{D(Tx, T^2 x, Tz)\}$$

$\forall x, y, z \in X$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in R: \alpha_1 + \alpha_2 + 3\alpha_3 < 1 - 3\alpha_4 - \alpha_5$ for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Proof: Let x_0 be an arbitrary point of X. Now we define a sequence $\{x_n\}$ by $x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots, x_n = Tx_{n-1}$. If for some $n \geq 0$, $x_n = x_{n+1}$, then x_n is a fixed point of T. Now assume that $x_n \neq x_{n+1} \forall\, n = 0, 1, 2, \ldots\ldots$ From the hypothesis we have Tsatisfies the condition

$$D(Tx,Ty,Tz) \leq \alpha_1 D(x,y,Tz) + \alpha_2 \max\{D(x,Ty,Tz), D(y,Tx,Tz)\} + \alpha_3 \max\{D(x,Ty,Tz), D(y,Tx,Tz)\} + \alpha_4 \max\{D(Ty,T^2x,Tz), D(x,T^2x,Tz)\} + \alpha_5 \max\{D(Tx,T^2x,Tz)\}.$$

From the above condition we achieve the result

$$D(Tx_{n-1},Tx_n,Tx_{n+1}) \leq \alpha_1 D(x_{n-1},x_n,x_{n+2}) + \alpha_2 \max\{D(x_{n-1},x_n,x_{n+2}), D(x_n,x_{n+1},x_{n+2})\} + \alpha_3 \max\{D(x_{n-1},x_{n+1},x_{n+2}), D(x_n,x_n,x_{n+2})\} + \alpha_4 \max\{D(x_{n+1},x_{n+1},x_{n+2}), D(x_{n-1},x_{n+1},x_{n+2})\} + \alpha_5 \max\{D(x_n,x_{n+1},x_{n+2})\}$$

or, $$D(x_n,x_{n+1},x_{n+2}) \leq \alpha_1 D(x_{n-1},x_n,x_{n+2}) + \alpha_2 D(x_{n-1},x_n,x_{n+2}) + \alpha_3 D(x_{n-1},x_n,x_{n+2}) + \alpha_4 D(x_{n+1},x_{n+1},x_{n+2}) + \alpha_5 D(x_n,x_{n+1},x_{n+2})$$
$$\leq \alpha_1 D(x_{n-1},x_n,x_{n+2}) + \alpha_2 D(x_{n-1},x_n,x_{n+2}) + \alpha_3[D(x_n,x_{n+1},x_{n+2}) + D(x_{n-1},x_n,x_{n+2}) + D(x_{n-1},x_{n+1},x_n)] + \alpha_4[D(x_n,x_{n+1},x_{n+2}) + D(x_{n+1},x_n,x_{n+2}) + D(x_{n+1},x_{n+1},x_n)] + \alpha_5 D(x_n,x_{n+1},x_{n+2})$$

or, $$[1-\alpha_3-2\alpha_4-\alpha_5]D(x_n,x_{n+1},x_{n+2}) \leq [\alpha_1+\alpha_2+\alpha_3]D(x_{n-1},x_n,x_{n+2}) + \alpha_3 D(x_{n-1},x_{n+1},x_n) + \alpha_4 D(x_{n+1},x_{n+1},x_n)$$
$$\leq [\alpha_1+\alpha_2+2\alpha_3+\alpha_4] \max[D(x_{n-1},x_n,x_{n+2}), D(x_{n-1},x_{n+1},x_n), D(x_{n+1},x_{n+1},x_n)]$$

$i.e.$ $D(x_n,x_{n+1},x_{n+2}) \leq p \max[D(x_{n-1},x_n,x_{n+2}), D(x_{n-1},x_{n+1},x_n), D(x_{n+1},x_{n+1},x_n)]$

Here $p = {\alpha_1+\alpha_2+2\alpha_3+\alpha_4}/{1-\alpha_3-2\alpha_4-\alpha_5} < 1$ due to $\alpha_1+\alpha_2+3\alpha_3 < 1-3\alpha_4-\alpha_5$

$i.e.$ $D(x_n,x_{n+1},x_{n+2}) \leq pD(x_{n-1},x_n,x_{n+2}), D(x_n,x_{n+1},x_{n+2}) \leq pD(x_{n-1},x_{n+1},x_n)$ and also

$D(x_n,x_{n+1},x_{n+2}) \leq pD(x_{n+1},x_{n+1},x_n)$ $i.e.$ $D(x_n,x_{n+1},x_{n+2}) \leq p^n D(x_0,x_1,x_3)$ similarly we achieve $D(x_n,x_{n+1},x_{n+2}) \leq p^n D(x_0,x_2,x_1)$ and $D(x_n,x_{n+1},x_{n+2}) \leq p^n D(x_2,x_2,x_1)$. This concludes the result that $D(x_n,x_{n+1},x_{n+2}) \leq p^n \max[D(x_0,x_1,x_3), D(x_0,x_2,x_1), D(x_2,x_2,x_1)]$ and hence $\{x_n\}$ is a Cauchy sequence because $p < 1$. On the other hand X is complete. So $\{x_n\}$ converges to a point $q \in X$. This implies that $\lim_{n\to\infty} D(T^{n+1}x_0, Tq, r) = 0$. But T is orbitally continuous $D(q,Tq,r) \leq D(q,Tq,T^{n+1}x_0) + D(q,T^{n+1}x_0,r) + D(T^{n+1}x_0,Tq,r)$ approaches to 0 as $n \to \infty$. Which leads us $d(q,Tq,r) = 0$ but $q \neq Tq$. Also $D(q,Tq,r) \neq 0$ for any r. Hence $q = Tq$. Hene q is a fixed point of T.

Corollary 2.3 If X be a complete D-metric space and T_1 and T_2 be self-mapping on X satisfy the criterion $\Delta_d\left(O_{T_1}(T_2x_0)\right) < \infty, T_1(X) \subseteq T_2(X)$, the pair (T_1,T_2) is D-compatible and T_2 is continuous, and for some $k \in [0,1)\ \forall\, u,v,w \in X$,

$D(T_1u, T_1v, T_1w) \leq k \max \begin{bmatrix} D(T_2u, T_2v, T_2w), D(T_1u, T_2u, T_2w), D(T_1y, T_2v, T_2w) \\ D(T_1u, T_2v, T_2w), D(T_1v, T_2u, T_2w) \end{bmatrix}$. Consequently, T_1 and T_2 share a unique common fixed point.

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