ON CONVOLUTABLE FRECHET SPACES OF DISTRIBUTIONS

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Abstract - *In this paper, many results of Fourier analysis which are known for convolutable Banach spaces of distributions (BCD-spaces) and Frechet spaces of distributions (FD-spaces) have been generalized to convolutable Frechet spaces of distributions (CFD-spaces). Also, we discuss the dual space of a CFD-space and obtain some useful results about homogeneous CFD-spaces.*

***Key Words*: Fourier analysis, Banach spaces, Frechet spaces, circle group, dual space and homogeneous space.**

1. INTRODUCTION

In [7] some results of Fourier analysis, which are known for $\boldsymbol{L^p}\,(\mathbf{1 \le p \le \infty})$, $\boldsymbol{C}$ and $\boldsymbol{M}$ etc., were obtained for convolutable Banach spaces of distributions (*BCD-spaces*). But those results cannot be applied to some important spaces like $\boldsymbol{C}^{\infty}$ (the space of all infinitely differentiable functions). Also, in [6] some results of Fourier analysis were obtained for Frechet spaces of distributions (*FD-spaces*). But those results cannot be applied to some important spaces like Hardy's spaces $\boldsymbol{H^p}\,(\mathbf{1 \le p < \infty})$. To overcome these deficiencies, in this paper we define the convolutable Frechet spaces of distributions (*CFD-spaces*).

In Section 2, we define *CFD-spaces* and state some preliminary results dealing with *CFD-spaces*. In Section 3, we define homogeneous *CFD-spaces* and obtain some important results about homogeneous *CFD-spaces*. In Section 4, we discuss the dual space of a *CFD-space* and obtain some useful results.

2. DEFINITIONS, NOTATIONS EXAMPLES AND PRELIMINARY RESULTS

We refer to [1], [5] and [8] for all the standard definitions, notations and assumptions. In particular, all our distributions are assumed to be defined on the circle group $\mathbf{G} = \mathbf{R}/2\pi\,\mathbf{Z}$, and the space of all distributions is denoted by $\boldsymbol{D}$.

2.1 Definition

A Frechet space $\boldsymbol{E}$ is called a convolutable Frechet space of distributions, briefly a *CFD-space*, if it can be continuously embedded in ($\boldsymbol{D}$, *strong**), and if, regarded as a subset of $\boldsymbol{D}$; it satisfies the following properties:

(2.1) $\mu \in \boldsymbol{M},\ f \in \boldsymbol{E} \Rightarrow \mu * f \in \boldsymbol{E}$, where $\boldsymbol{M}$ denotes the set of all (Radon) measures.

(2.2) $\boldsymbol{C}^{\infty} \cap \boldsymbol{E}$ is a closed subspace of $\boldsymbol{C}^{\infty}$.

It is obvious that every *BCD-space* is a *CFD-space* (see the definition of *BCD*-space in [7]). But $\boldsymbol{C}^{\infty}$ is a *CFD-space* which is not a *BCD-space* as it is not a Banach space.

Throughout the paper, $\boldsymbol{E}$, if not specified, will denote a *CFD-space* and $\boldsymbol{E}^*$ will denote its *strong** dual (see [8], Ch. 10).

2.2. We now give an example of a non-empty Frechet space $\boldsymbol{E}$ continuously embedded in $\boldsymbol{D}$ which satisfies the assumption (2.1) but not (2.2).

Let $\boldsymbol{E}$ be the set of all $f \in \boldsymbol{C}^{\infty}$ such that $\|f\|_{E} < \infty$ where

$$\|f\|_{\mathrm{E}} = \sum_{k=0}^{\infty} \frac{\|D^k f\|_{\infty}}{(k+1)}$$

Then $\boldsymbol{E}$ is the required space as every Banach space is a Frechet space (see [7]).

2.3. Now, we give an example of a non-empty Frechet space $\boldsymbol{E}$ continuously embedded in $\boldsymbol{D}$, such that (2.2) is satisfied but not (2.1).

Let $M_d(\mathrm{G})$ denote the set of purely discontinuous measures on $\mathbf{G}$ i.e., the set of all those measures μ on $\mathbf{G}$ for which there exists a countable subset $\boldsymbol{A}$ of $\mathbf{G}$ such that $|\mu|(A^C) = 0$. Then $M_d(\mathrm{G})$ is the required space (see [7]).

The above examples show the independence of both the assumptions taken in the definition of a *CFD*-space.

2.4. Theorem. Let $\boldsymbol{E}$ be a *CFD-space*. Then the transformation $S : \boldsymbol{M} \times \boldsymbol{E} \to \boldsymbol{E}$ defined by

$S(\mu, f) = \mu * f$ for each $\mu \in \boldsymbol{M}$ and $f \in \boldsymbol{E}$,

is continuous on $\boldsymbol{M} \times \boldsymbol{E}$. Further, for each continuous seminorm p on $\boldsymbol{E}$, there exists a continuous seminorm q on $\boldsymbol{E}$ such that $p(\mu * f) \le \|\mu\|_1\, q(f)$ for each $\mu \in \boldsymbol{M}$ and for each $f \in \boldsymbol{E}$.

Proof: Fixing μ in $\boldsymbol{M}$, consider the mapping $T_1 : \boldsymbol{E} \to \boldsymbol{E}$ defined by $T_1 f = \mu * f$ for all $f \in \boldsymbol{E}$. Now, $\boldsymbol{E}$ is continuously embedded in $\boldsymbol{D}$ and T_1 is continuous on $\boldsymbol{D}$. So, using the closed graph theorem, we can show that T_1 is a continuous linear operator on $\boldsymbol{E}$. Similarly, if we define the mapping $T_2 : \boldsymbol{M} \to \boldsymbol{E}$ by $T_2 \mu = \mu * f$ for all $\mu \in \boldsymbol{E}$, after fixing f in $\boldsymbol{E}$, then T_2 turns out to be linear and continuous. Thus the transformation $S : \boldsymbol{M} \times \boldsymbol{E} \to \boldsymbol{E}$, defined by $S(\mu, f) = \mu * f$ is bilinear and separately continuous; and hence by ([5],p.52), S is (jointly) continuous on $\boldsymbol{M} \times \boldsymbol{E}$.

Further, by ([2], Exercise1, p.363), for each continuous seminorm p on $\boldsymbol{E}$, there exists a continuous seminorm q on $\boldsymbol{E}$ such that

$p(\mu * f) \le \|\mu\|_1\, q(f)$ for each $\mu \in \boldsymbol{M}$ and for each $f \in \boldsymbol{E}$.

Taking μ as the nth Fejer's kernel in the above theorem, we obtain the following.

2.5. Corollary. If $\boldsymbol{E}$ is a *CFD-space*, then for each continuous seminorm p on $\boldsymbol{E}$, there exists a continuous seminorm q on $\boldsymbol{E}$ such that

$$p(\sigma_n(f)) \le q(f)$$

where $\sigma_n f = F_n * f$ and F_n denotes the nth Fejer's kernel.

Taking μ as the Dirac measure at the point x in the above theorem, we obtain the following.

2.6. Corollary. If $\boldsymbol{E}$ is a *CFD-space*, then $\boldsymbol{E}$ is translation invariant and for each continuous seminorm p on $\boldsymbol{E}$ there exists a continuous seminorm q on $\boldsymbol{E}$ such that $p(T_x f) \le q(f)$ for each $x \in \mathbf{G}$ and for each $f \in \boldsymbol{E}$.

where $T_x f$ denotes the translation of f by x.

Also, it shows that $\{T_x : x \in \mathbf{G}\}$ is an equicontinuous family of translation operators on $\boldsymbol{E}$.

Using the closed graph theorem, and the fact that $\boldsymbol{C}^\infty \cap \boldsymbol{E}$ is closed in $\boldsymbol{C}^\infty$, we can easily prove the following.

2.7. Lemma. The inclusion map $\boldsymbol{i} : \boldsymbol{C}^\infty \cap \boldsymbol{E} \to \boldsymbol{E}$ is continuous, where $\boldsymbol{C}^\infty \cap \boldsymbol{E}$ has the relative topology of $\boldsymbol{C}^\infty$.

2.8. Theorem. A necessary and sufficient conditions for $\boldsymbol{P} \cap \boldsymbol{E}$ to be dense in $\boldsymbol{E}$ is that $\boldsymbol{C}^\infty \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$, where $\boldsymbol{P}$ is the set of all trigonometric polynomials.

Proof: One part is obvious as $\boldsymbol{P} \subset \boldsymbol{C}^\infty$.

Conversely, suppose $\boldsymbol{C}^\infty \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$. Let d be the metric on $\boldsymbol{E}$ induced by

$$\|f\|_E = \sum_{k=1}^{\infty} \frac{2^{-k}\, p_k(f)}{1 + p_k(f)}$$

where $\{p_k\}_{k=1}^{\infty}$ is a countable family of seminorms on $\boldsymbol{E}$ which defines the locally convex topology of $\boldsymbol{E}$. Given f in $\boldsymbol{E}$ and $\varepsilon > 0$, there exists $\boldsymbol{u} \in \boldsymbol{C}^\infty \cap \boldsymbol{E}$ such that $d(f, u) < \varepsilon / 2$.

Now,

$\sigma_n u \to u$ in $\boldsymbol{C}^\infty$ as $n \to \infty$ and $\boldsymbol{\sigma_n u} \in \boldsymbol{C}^\infty \cap \boldsymbol{E}$ $\forall \boldsymbol{n} \in \boldsymbol{Z}$.

Since $\boldsymbol{i} : \boldsymbol{C}^\infty \cap \boldsymbol{E} \to \boldsymbol{E}$ is continuous by above lemma, $\sigma_n u \to u$ in $\boldsymbol{C}^\infty \cap \boldsymbol{E}$ with relative topology of $\boldsymbol{E}$. So corresponding to $\varepsilon > 0$, there exists $N > 0$ such that $d(\sigma_n u, u) < \varepsilon / 2$ for all $n \ge N$.

Hence,

$d(\sigma_n u, f) \le d(f, u) + d(\sigma_n u, u) < \varepsilon$ for all $n \ge N$.

Therefore $\boldsymbol{P} \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$.

3. HOMOGENOUS CFD-SPACES

Homogenous Banach subspaces of $\boldsymbol{L}^1$ defined on the circle group $\mathbf{G}$ are discussed in [3] and many results of Fourier analysis on these spaces are generalized to homogeneous *BCD-spaces* in [7] and to homogeneous *FD-spaces* in [6]. We define homogeneous *CFD-space* as follows.

3.1. Definition. A CFD-*space* $\boldsymbol{E}$ is said to be homogenous if $x \to x_0$ in $\mathbf{G}$ implies $T_x f \to T_{x_0} f$ in $\boldsymbol{E}$ for each $f \in \boldsymbol{E}$.

3.2. Theorem. Every homogeneous *FD-space* is a homogeneous *CFD-space*.

Proof: Let $\boldsymbol{E}$ be a homogeneous *FD-space* From the definition of a *FD-space* and Theorem 5.5 (iii) of [6], $\mu \in \boldsymbol{M}, f \in \boldsymbol{E} \Rightarrow \mu * f \in \boldsymbol{E}$ and $\boldsymbol{C}^\infty \cap \boldsymbol{E} = \boldsymbol{C}^\infty$ is closed in $\boldsymbol{C}^\infty$ as $\boldsymbol{C}^\infty \subset \boldsymbol{E}$. Thus $\boldsymbol{E}$ is a *CFD-space*.

But converse of the above theorem is not true as shown in the following example

3.3. *CFD-spaces* which are not *FD-spaces*

For $p \geq 1$, consider the Hardy space H^p defined by

$\boldsymbol{H^p = \{f \in L^p : \hat{f}(n) = 0 \text{ for } n < 0\}}$.

Clearly H^p is a homogeneous *BCD-space* for $1 \leq p < \infty$ ([7], 2.2) and we know that every *BCD-space* is a *CFD-space*, therefore H^p is a homogeneous *CFD-space* for $1 \leq p < \infty$.

But H^p is not a *FD-space* as C^∞ is not contained in H^p.

3.4. Example of a non-homogeneous *CFD*-spaces is $\boldsymbol{M}$.

The proof of the following lemma and theorem are just like the ones given in 5.2 and 5.3 of [6].

3.5. Lemma. Let $\boldsymbol{E}$ be a Frechet space, ϕ a continuous $\boldsymbol{E}$-valued function on $\mathbf{G}$ and $\{K_n\}_{n=1}^\infty$ a summability kernel (approximate identity), then

$$\lim_{n\to\infty} \frac{1}{2\pi} \int K_n(t)\phi(t)\,dt = \phi(0)$$

where the integral is taken over $\mathbf{G}$.

3.6. Theorem. If $\boldsymbol{E}$ is a homogeneous *CFD-space*, then for each $f \in \boldsymbol{E}$, $\sigma_n f \to f$ in $\boldsymbol{E}$ as $n \to \infty$, where $\sigma_n f$ denotes the n-th Cesaro sum of the Fourier series of f.

3.7. Theorem. A *CFD-space* $\boldsymbol{E}$ is homogeneous if and only if $\boldsymbol{P} \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$.

Proof: Let $\boldsymbol{E}$ be homogeneous. Then by the above theorem, for each $f \in \boldsymbol{E}$, $\sigma_n f \to f$ in $\boldsymbol{E}$ as $n \to \infty$. Hence $\boldsymbol{P} \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$.

Conversely, suppose $\boldsymbol{P} \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$. Let $f \in \boldsymbol{E}$ and $\varepsilon > 0$. Since $\{T_x : x \in \mathbf{G}\}$ is an equicontinuous family of translation operators on $\boldsymbol{E}$, by Corollary 2.6., there exists a $\delta > 0$ such that

(3.1) $d(f, g) < \delta \Rightarrow d(T_x f, T_x g) < \varepsilon/3$.

Since $C^\infty \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$ (see Theorem 2.8), we may fix a function g in $C^\infty \cap \boldsymbol{E}$ such that $d(f, g) < \delta$.

Now $T_x g \to T_{x_0} g$ in C^∞ as $x \to x_0$ and therefore by Lemma 2.7, $T_x g \to T_{x_0} g$ in $\boldsymbol{E}$ as $x \to x_0$. So $\exists\ \eta > 0$ such that

(3.2) $|x - x_0| < \eta \Rightarrow d(T_x f, T_{x_0} g) < \varepsilon/3$.

Also, $d(T_x f, T_{x_0} f)$

$$\leq d(T_x f, T_x g) + d(T_x g, T_{x_0} g) + d(T_{x_0} f, T_{x_0} g).$$

Thus by (3.1) and (3.2),

$$d(T_x f, T_{x_0} f) < \varepsilon \text{ for } |x - x_0| < \eta.$$

Hence, $T_x f \to T_{x_0} f$ in $\boldsymbol{E}$ as $x \to x_0$, which shows that $\boldsymbol{E}$ is homogeneous.

Using Theorem 2.8, Theorem 3.6 and the above theorem, we get the following.

3.8. Corollary. Let $\boldsymbol{E}$ be a *CFD-space.* Then the following four results are equivalent:

(i) $\boldsymbol{E}$ is homogeneous;
(ii) $\boldsymbol{P} \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$;
(iii) $C^\infty \cap \boldsymbol{E}$ is dense in $\boldsymbol{E}$;
(iv) $\forall\ f \in \boldsymbol{E}$, $\sigma_n f \to f$ in $\boldsymbol{E}$ as $n \to \infty$.

4. DUAL SPACES

If $\boldsymbol{E}$ is a Frechet space, then $\boldsymbol{E}^*$ need not be a Frechet space. So, if $\boldsymbol{E}$ is a *CFD-space*, we cannot say that $\boldsymbol{E}^*$ is also a *CFD-space.* However we can still embed $\boldsymbol{E}^*$ in $\boldsymbol{D}$ and treat the elements of $\boldsymbol{E}^*$ as distributions. For this we have to make a one to one correspondence between the elements of $\boldsymbol{E}^*$ and the elements of $\boldsymbol{D}$. The task would have been easier if C^∞ had been contained in $\boldsymbol{E}$, but it is not. So, we define the following sets

(4.1) $S = \{n \in Z : e_n \in \boldsymbol{E}\}, S_N = \{n \in S : |n| \leq N\}$ and

The proof of the following four results is similar to the corresponding results in [7].

4.1. Lemma. $e_n \in \boldsymbol{E}$ if and only if $\hat{f}(n) \neq 0$ for some $f \in \boldsymbol{E}$.

Using the definition of S and the above lemma we immediately get the following.

4.2. $n \notin S \Rightarrow \hat{f}(n) = 0 \quad \forall\ f \in \boldsymbol{E}$.

From now onwards, by $\sum a_k$ we shall mean $\sum_{k=1}^{\infty} a_k$ and, by $\sum_S a_k$ we shall mean $\lim_{N\to\infty} \sum_{k\in S_N} a_k$.

4.3. Lemma. For $u = \sum \hat{u}(k) e_k \in \boldsymbol{C}^\infty$, define

$$Pu = \sum_S \hat{u}(k) e_k .$$

Then P is a continuous projection of $\boldsymbol{C}^\infty$ onto $\boldsymbol{C}^\infty \cap \boldsymbol{E}$.

Let us define $\boldsymbol{j} : \boldsymbol{E}^* \to \boldsymbol{D}$ by

$\boldsymbol{j(F)(u) = F(Pu)}$ for all u in $\boldsymbol{C}^\infty$ and for all F in $\boldsymbol{E}^*$

where P is the projection defined in the above lemma.

Now, we can claim the following (See [7], Theorem 4.1)

4.4. Theorem. Let $\boldsymbol{E}$ be a *CFD*-space. Then $\boldsymbol{E}^*$ can be continuously embedded into $\boldsymbol{D}$ through the mapping j and as a subset of $\boldsymbol{D}$, it satisfy the following properties:

(4.2) $\mu \in \boldsymbol{M},\ f \in \boldsymbol{E}^* \Rightarrow \mu * f \in \boldsymbol{E}^*$

(4.3) $\boldsymbol{C}^\infty \cap \boldsymbol{E}^*$ is a closed subspace of $\boldsymbol{C}^\infty$.

4.5. Theorem. Let $\boldsymbol{E}$ be a homogeneous *CFD-space* and $F \in \boldsymbol{E}^*$. Then for each $f \in \boldsymbol{E}$, the function $g(x)$, defined by $g(x) = F(T_x f)$ for all $x \in \mathbf{G}$, is continuous on $\mathbf{G}$ and generates the distribution $F * \check{f}$. Moreover, there exists a continuous seminorm q on $\boldsymbol{E}$ such that

$$\| F * \check{f} \|_\infty \le q(f) \quad \forall\ f \in \boldsymbol{E}.$$

Proof: As $\boldsymbol{E}$ is homogeneous,

$x \to x_0$ in $\mathbf{G} \Rightarrow \mathbf{T}_x f \to T_{x_0} f$ in $\boldsymbol{E}$.

Also, F is continuous on $\boldsymbol{E}$. Hence $g(x)$ is continuous on $\mathbf{G}$ for each $f \in \boldsymbol{E}$. Now, by the Corollary 2.6., the family $\{T_x : x \in \mathbf{G}\}$ of translation operators on $\boldsymbol{E}$ is equicontinuous, therefore $\{FoT_x : x \in \mathbf{G}\}$ is also equicontinuous. So, using Theorem 9.5.3 ([4], p.203), there exists a continuous seminorm q on $\boldsymbol{E}$ such that $|g(x)| = |F(T_x f)| \le q(f)\ \forall f \in \boldsymbol{E}$ and $\forall x \in \mathbf{G}$. Hence,

(4.4) $\| g \|_\infty \le q(f) \quad \forall\ f \in \boldsymbol{E}$.

Now our aim is to prove that $g(x)$ generates the distribution $F * \check{f}$. Define $g_n(x) = F(\sigma_n T_x f)$ for every x in $\mathbf{G}$. Since $\boldsymbol{E}$ is homogeneous, by Theorem 3.6,

$$\sigma_n T_x f \to T_x f \text{ in } \boldsymbol{E}.$$

Consequently,

(4.5) $g_n(x) = F(\sigma_n T_x f) \to F(T_x f) = g(x)$ as $n \to \infty$, for every x.

Now, for every f in $\boldsymbol{E}$,

$\lim_{n\to\infty} \sigma_n F(f) = \lim_{n\to\infty} F(\sigma_n f) = F(f)$ as $\boldsymbol{E}$ is homogeneous.

By ([8], Theorem 9.3.4.), $\{\sigma_n F\}$ is an equicontinuous family of continuous linear functional on $\boldsymbol{E}$ which converges pointwise to F in $\boldsymbol{E}^*$. Therefore arguing just as above, we can find a continuous seminorm q' in $\boldsymbol{E}$ such that for all positive integers n,

$$|\sigma_n (F * \check{f})(x)| = |\sigma_n F(T_x f)| \le q'(f) \ \forall x \in \mathbf{G}, \forall f \in \boldsymbol{E}$$

which gives that $\| \sigma_n (F * \check{f}) \|_\infty = O(1)$.

Therefore $F * \check{f} \in \boldsymbol{L}^\infty$, and, for a.e. x,

$$g_n(x) = \sigma_n (F * \check{f})(x) \to F * \check{f}(x) \text{ as } n \to \infty.$$

Now, it follows from (4.5) that

$g(x) = F * \check{f}(x)$ for a.e. x

Hence $g(x)$ generates the distribution $F * \check{f}$,

i.e., $g = F * \check{f}$ in the sense of distribution.

Now, from (4.4), $\| F * \check{f} \|_\infty \le q(f)$ for all f in $\boldsymbol{E}$.

4.6. Theorem. A *CFD-space* $\boldsymbol{E}$ is homogenous if and only if for every $f \in \boldsymbol{E}$ and $F \in \boldsymbol{E}^*$, the series

(4.6) $$\sum_{-\infty}^{\infty} \hat{F}(n)\, \hat{f}(-n)$$

is (C, 1)-summable to $F(f)$.

Proof: One part is clear from Theorem 3.6 and the fact that $F(\sigma_n f)$ is the n-th Cesaro sum of (4.6). Conversely, suppose that the series (4.6) is (C, 1)-summable to $F(f)$ for every f in $\boldsymbol{E}$ and F in $\boldsymbol{E}^*$. Then

$$\lim_{n\to\infty} F(\sigma_n f) = F(f) \quad \forall\ f \in \boldsymbol{E} \text{ and } \forall\ F \in \boldsymbol{E}^*.$$

This shows that $\sigma_n f$ converges weakly to f in $\boldsymbol{E}$ for every f in $\boldsymbol{E}$. Thus $\boldsymbol{P} \cap \boldsymbol{E}$ is weakly dense in $\boldsymbol{E}$. Since $\boldsymbol{P} \cap \boldsymbol{E}$ is convex, $\boldsymbol{P} \cap \boldsymbol{E}$ is strongly dense in $\boldsymbol{E}$ by ([5], p.66). Hence by Theorem 3.7, $\boldsymbol{E}$ is homogenous.

5. CONCLUSIONS

In this paper we define Convolutable Frechet Spaces of Distributions (briefly written as *CFD-spaces*) and generalize the previously known results to *CFD-spaces*. We have used various results and techniques of Functional analysis to obtain these results. The results obtained will be useful for further analysis in the field of Fourier analysis and Functional analysis.

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